

# THE MANIN-PEYRE FORMULA FOR A CERTAIN BIPROJECTIVE THREEFOLD

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ABSTRACT. The conjectures of Manin and Peyre are confirmed for a certain threefold.

## 1. INTRODUCTION

**1.1. The main result.** In a recent memoir [4] we confirmed the predictions of Manin and Peyre for the distribution of rational points on the cubic fourfold in  $\mathbb{P}^5$  defined by

$$(1.1) \quad x_1y_2y_3 + x_2y_1y_3 + x_3y_1y_2 = 0.$$

Here, we continue our study of this equation, now viewing the polynomial on the left hand side as a linear form in  $\mathbf{x} = (x_1, x_2, x_3)$  and a quadratic form in  $\mathbf{y} = (y_1, y_2, y_3)$ . With  $\mathbf{x}$  and  $\mathbf{y}$  interpreted as homogeneous coordinates, the equation (1.1) defines a variety  $V$  in  $\mathbb{P}^2 \times \mathbb{P}^2$ . For a  $\mathbb{Q}$ -rational point on  $V$  there are representatives  $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^3$  with  $(x_1; x_2; x_3) = (y_1; y_2; y_3) = 1$ , both unique up to sign. An anticanonical height function on  $V$  is then given by

$$(1.2) \quad H(\mathbf{x}, \mathbf{y}) = \max_{1 \leq i, j \leq 3} |x_i|^2 |y_j|.$$

Rational points on  $V$  ordered with respect to this height accumulate on the subvariety cut out from  $V$  by the additional equation  $x_1x_2x_3y_1y_2y_3 = 0$ . To see this, note that the choices  $\mathbf{x} = (0, 1, 1)$  and  $\mathbf{y} = (y_1, y_2, -y_2)$  with  $(y_1; y_2) = 1$  produce more than  $B^2$  rational points of height at most  $B$  on this subvariety, while on the Zariski-open subset  $V^\circ$  of  $V$  where  $x_1x_2x_3y_1y_2y_3 \neq 0$  the rational points are much sparser. This is a consequence of the following asymptotic formula.

**Theorem 1.1.** *Let  $N(B)$  denote the number of rational points on  $V^\circ$  with height not exceeding  $B$ , and let*

$$(1.3) \quad C = \prod_p \left(1 - \frac{1}{p}\right)^5 \left(1 + \frac{5}{p} + \frac{5}{p^2} + \frac{1}{p^3}\right).$$

*Then*

$$(1.4) \quad N(B) = \frac{\pi^2 - 3 + 24 \log 2}{144} CB(\log B)^4 + O(B(\log B)^{4 - \frac{1}{480}}).$$

Hitherto it was only known [2] that  $N(B) \asymp B(\log B)^4$ . No effort has been made to optimize the error term in (1.4). With more work it is possible to show that there is a polynomial  $P$  of degree four and a positive number  $\delta$  with the property that

$$(1.5) \quad N(B) = BP(\log B) + O(B^{1-\delta}),$$

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but in order to keep the paper at reasonable length we have to content ourselves with a detailed proof of (1.4). The reader is referred to Section 1.2 for a brief summary of refinements needed to establish (1.5).

The shape of the asymptotic formula (1.4) is in line with a general conjecture of Manin (see [10]) concerning the distribution of rational points on smooth Fano varieties. However,  $V$  has three singularities located at  $x_i = x_j = y_i = y_j$  for  $1 \leq i < j \leq 3$ . A resolution has been obtained in [4, Theorem 4]: Let  $X \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$  be the triprojective variety defined in trihomogeneous coordinates  $(\mathbf{x}, \mathbf{y}, \mathbf{z})$  by

$$(1.6) \quad x_1 z_1 + x_2 z_2 + x_3 z_3 = 0 \quad \text{and} \quad y_1 z_1 = y_2 z_2 = y_3 z_3.$$

Then the restriction to  $X$  of the projection  $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$  onto the first two factors is a *crepant* resolution of  $V$ , and one has  $\text{rk Pic}(X) = 5$ . In this situation, Batyrev and Tschinkel [1] predict that  $N(B)/(B(\log B)^4)$  tends to a limit as  $B \rightarrow \infty$ , and Peyre [19] suggested a formula for this limit. At the end of this paper, in Sections 7 and 8, we show that our findings in Theorem 1.1 agree with Peyre’s formula. Notice in particular that the  $p$ -adic factor of the constant  $C$  in (1.3) is the  $p$ -adic density of the universal torsor over  $X$ , which in our case is a  $\mathbb{G}_m^5$ -torsor over a  $\mathbb{P}^1$ -bundle over a del Pezzo surface of degree six.

The Manin-Peyre conjectures for the distribution of rational points on algebraic varieties have received considerable attention in recent years. Powerful techniques are available for surfaces (see e.g. the references in [7]). Moreover, the circle method typically produces asymptotic relations that confirm the conjectures, but this requires the dimension be large in terms of the degree. If the variety carries additional structure, further tools can be brought into play. For example, when the variety is an equivariant compactification of certain linear algebraic groups, Tschinkel and his collaborators applied adelic Fourier analysis to prove Manin’s conjecture in some generality (see e.g. [9, 24]).

The variety under consideration is not covered by the cases just described. Definitive results for Fano threefolds are very rare besides the remarkable paper of de la Bretèche [6] on the Segre cubic  $(x_1 + \dots + x_5)^3 = x_1^3 + \dots + x_5^3$ . Le Boudec [5] determined the order of magnitude for the number of rational points of bounded height on the biprojective threefold  $x_1 y_1^2 + x_2 y_2^2 + x_3 y_3^2 = 0$ , and we agree with him that a refinement to an asymptotic formula seems “far out of reach”.

Although we are concerned here with just one concrete example, the methods that underpin the the proof of Theorem 1.1 are by no means restricted to the case at hand. The family of varieties defined by

$$\frac{x_1}{y_1} + \frac{x_2}{y_2} + \dots + \frac{x_n}{y_n} = 0$$

springs to mind of which we treat the case  $n = 3$  here. Larger  $n$  should be within reach for the techniques described herein, and we intend to return to the theme in a broader setting in due course. We hope that the present example spurs further work on higher dimensional cases of the Manin-Peyre formula.

**1.2. The methods.** The ideas behind the proof of Theorem 1.1 have some similarity with our earlier work [4] where the cubic form (1.1) was studied as a fourfold in  $\mathbb{P}^5$ . Yet, there are several fundamental differences. The initial step is the same as in [4]. An elementary argument transfers the original counting problem to one on the universal torsor. The latter is given by

$$(1.7) \quad u_1 v_1 + u_2 v_2 + u_3 v_3 = 0,$$

and it is the simple shape of this bilinear equation what makes the variety  $V$  accessible to analytic methods. It is typical that box-like conditions on the original equation transform to regions with hyperbolic spikes on the torsor. In the situation considered here, the anticanonical height function (1.2) involves a product, resulting in very narrow spikes where integral points are difficult to count. This forced us to waive the strategy followed in [4] where the solutions of (1.7) were parametrized in the obvious way, leading to a hyperbolic lattice point problem that was then approached through multiple Dirichlet series. Instead, we use Fourier analysis directly to count points on the torsor. At

the core of the method, we then require an asymptotic formula for the number  $N_{\mathbf{r}}(\mathbf{X}, \mathbf{Y})$  of solutions to the equation

$$(1.8) \quad r_1 x_1 y_1 + r_2 x_2 y_2 + r_3 x_3 y_3 = 0$$

in  $\mathbf{x} \in \mathbb{Z}^3$ ,  $\mathbf{y} \in \mathbb{Z}^3$  within the region

$$\mathcal{B}(\mathbf{X}, \mathbf{Y}) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \tfrac{1}{2}X_j < |x_j| \leq X_j, \tfrac{1}{2}Y_j < |y_j| \leq Y_j, 1 \leq j \leq 3\}.$$

Here  $\mathbf{r} = (r_1, r_2, r_3) \in \mathbb{N}^3$  are given coefficients,  $\mathbf{X}, \mathbf{Y}$  are triples of positive real numbers, and we need the asymptotic formula uniformly with respect to  $\mathbf{r}$ . Note that  $\mathcal{B}(\mathbf{X}, \mathbf{Y})$  consists of  $2^6$  possibly very lopsided boxes. Nonetheless this counting problem is within the competence of the circle method. The following proposition delivers the desired asymptotics, and we shall save a small power of the smallest side of the box. The asymptotic formula involves the singular series

$$(1.9) \quad \mathcal{E}_{\mathbf{r}} = \sum_{q=1}^{\infty} \frac{\varphi(q)(q; r_1)(q; r_2)(q; r_3)}{q^3}$$

and the singular integral

$$(1.10) \quad \mathcal{I}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) = \int_{\mathbb{R}} \int_{\mathcal{B}(\mathbf{X}, \mathbf{Y})} e(\alpha(r_1 x_1 y_1 + r_2 x_2 y_2 + r_3 x_3 y_3)) d(\mathbf{x}, \mathbf{y}) d\alpha.$$

**Proposition 1.2.** *Let  $X_1, X_2, X_3, Y_1, Y_2, Y_3 \geq 1$  and  $r_1, r_2, r_3 \in \mathbb{N}$ . Then*

$$N_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) = \mathcal{E}_{\mathbf{r}} \mathcal{I}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) + \Theta_{\mathbf{r}}(\mathbf{X}, \mathbf{Y})$$

where for any fixed positive value of  $\varepsilon$  one has

$$\Theta_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) \ll \frac{(r_1 X_1 Y_1 \cdot r_2 X_2 Y_2 \cdot r_3 X_3 Y_3)^{1+\varepsilon}}{\max(r_1 X_1 Y_1, r_2 X_2 Y_2, r_3 X_3 Y_3) \min(X_1, X_2, X_3, Y_1, Y_2, Y_3)^{1/6}}.$$

The implicit constant depends at most on  $\varepsilon$ .

This asymptotic formula is what the circle method predicts, although our proof uses a different argument in which the key ingredient is a non-trivial bound for Kloosterman sums. The dependence on  $r_1, r_2, r_3$  in the error term  $\Theta_{\mathbf{r}}(\mathbf{X}, \mathbf{Y})$  can be improved.

With this result in hand, one can apply a simple version of the patchwork method developed in [3]. For some small  $\delta > 0$ , we glue together the contributions from boxes where

$$\min(X_1, X_2, X_3, Y_1, Y_2, Y_3) \geq \max(X_1, X_2, X_3, Y_1, Y_2, Y_3)^{\delta}.$$

This keeps us away from the spikes, and then we send  $\delta$  to 0 at an appropriate speed. In this way, the ideas underpinning the proof [3, Lemma 2.8] deliver the conclusions recorded in Theorem 1.1. The factorization of the leading constant in (1.4) is imported from similar properties of the main term in the asymptotics announced in Proposition 1.2. We are fortunate that all local factors can be computed explicitly.

Once Theorem 1.1 is established, we turn to the task of comparing the result with the predictions made by Manin and Peyre. In this endeavour, we need to compute some invariants of the crepant resolution  $X$ , cf. (1.6). To compute Peyre's alpha invariant  $\alpha(X)$ , we endow the invertible  $\mathcal{O}_X$ -modules with  $\mathbb{G}_m^3$ -linearizations compatible with a (non-faithful)  $\mathbb{G}_m^3$ -action on  $X$  induced by the embedding of  $X$  in  $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ . This implies (see Lemma 7.2) that any effective divisor on  $X$  is linearly equivalent to a  $\mathbb{G}_m^3$ -invariant effective divisor. It is then not too hard to show that the pseudo-effective cone in  $\text{Pic } X$  is spanned by nine  $\mathbb{G}_m^3$ -invariant prime divisors and to calculate  $\alpha(X)$ .

To compute the adelic volume in Peyre's constant, we proceed as in [4] and relate the volume forms on the non-singular locus of  $V$  to Poincaré residues of meromorphic forms on  $\mathbb{P}^2 \times \mathbb{P}^2$  with poles along  $V$ . We then see that the Euler product in Theorem 1.1 is the expected one, and also that the product of the  $\alpha$ -invariant and the archimedean volume agrees with the first factor on the right hand side of (1.4).

As we have pointed out before, our arguments can be refined further. Indeed, one may develop Proposition 1.2 so as to cover the case where some of the variables in (1.8) are fixed, and equipped with this, one can use the machinery from [3] in full, to cope with the cuspidal portion of the counting more precisely. This analysis provides error terms that save a power of the largest side of the box. Then, by the main result of [3] one obtains (1.5). Also, one may sum by parts to obtain an analytic continuation of the 6-fold Dirichlet series

$$(1.11) \quad \sum' \frac{1}{|x_1|^{s_1} |x_2|^{s_2} |x_3|^{s_3} |y_1|^{t_1} |y_2|^{t_2} |y_3|^{t_3}},$$

where the prime indicates summation over  $(\mathbf{x}, \mathbf{y}) \in (\mathbb{Z} \setminus \{0\})^6$  satisfying (1.8). If one specialises to  $r_1 = r_2 = r_3 = 1$  and then restricts to the diagonal  $s_1 = s_2 = s_3, t_1 = t_2 = t_3$ , this series is essentially a minimal parabolic Eisenstein series for  $\mathrm{SL}_3(\mathbb{Z})$ , see [8]. The series (1.11) is a far-reaching generalization of this well-understood Eisenstein series that, apparently, does no longer memorize the group theoretic information carried by its ancestor. Perhaps this is the reason for the considerable complexity of our analysis of the threefold defined by (1.1).

Details of the arguments outlined in the preceding paragraph will not be worked out in this paper. Armed with this refinement of Proposition 1.2 it would be straightforward but elaborate to do so.

*Notation.* Most of the notation used in this paper is either standard or otherwise explained at the appropriate stage of the argument. However, traditional notation in the various branches in mathematics on which our work is built resulted in clashes, and the desire for entire consistency in this respect turned out to be impracticable. The following guide may help the reader to clarify the symbolism in the work to follow.

Throughout, we apply the following convention concerning the letter  $\varepsilon$ . Whenever  $\varepsilon$  occurs in a statement, may it be explicitly or implicitly, then it is asserted that the statement is true for any fixed positive real number in the role of  $\varepsilon$ . Constants implicit in the use of Landau's or Vinogradov's well-known symbols may then depend on the value assigned to  $\varepsilon$ . Note that this allows us to conclude from the inequalities  $A \ll X^\varepsilon$  and  $B \ll X^\varepsilon$  that  $AB \ll X^\varepsilon$ , for example.

Frequently, we use vector notation  $\mathbf{x} = (x_1, \dots, x_n)$  where the underlying field and the dimension  $n$  is usually clear from the context. If the coordinates  $x_j$  are complex numbers, we write

$$(1.12) \quad |\mathbf{x}| = \max_j |x_j|, \quad |\mathbf{x}|_1 = \sum_{j=1}^n |x_j|.$$

Further, when  $\mathbf{x} \in \mathbb{R}^n$  and  $\mathbf{a} \in \mathbb{C}^n$  we write

$$(1.13) \quad \mathbf{x}^{\mathbf{a}} = |x_1|^{a_1} |x_2|^{a_2} \dots |x_n|^{a_n}.$$

We will often have to integrate over vertical lines in the complex plane. In this context, when  $c$  is a real number, the parametrized line  $(-\infty, \infty) \rightarrow \mathbb{C}, t \mapsto c + it$  is denoted by  $(c)$ .

The number of divisors of the natural number  $n$  is  $\tau(n)$ . The Möbius function is denoted by  $\mu(n)$ , and  $\varphi(n)$  is Euler's totient. The greatest common divisor of the non-zero integers  $a_1, \dots, a_n$  is denoted by  $(a_1; \dots; a_n)$ , and their least common multiple is  $[a_1; \dots; a_n]$ . When  $f : \mathbb{N} \rightarrow \mathbb{C}$  is an arithmetical function and  $\mathbf{a} \in \mathbb{N}^n$ , we put, by slight abuse of notation,

$$f(\mathbf{a}) = f(a_1)f(a_2) \cdots f(a_n).$$

We put  $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$ . The cardinality of a finite set  $\mathcal{S}$  is  $|\mathcal{S}|$ . For a real number  $\theta$  we put  $e(\theta) = \exp(2\pi i \theta)$ .

## 2. AUXILIARY TOOLS

**2.1. Smoothing.** To accelerate convergence of certain integrals, we need a smooth approximation to the characteristic function of the unit interval  $f_0 : [0, \infty) \rightarrow [0, 1]$ . This is achieved via a conventional

convolution argument. For  $0 < \Delta < 1$  one can find a smooth function  $\varrho_\Delta : [0, \infty) \rightarrow [0, \infty)$  with

$$(2.1) \quad \text{supp}(\varrho_\Delta) \subset (1, 1 + \Delta) \quad \text{and} \quad \int_0^\infty \varrho_\Delta(x) \frac{dx}{x} = 1$$

such that  $\varrho_\Delta^{(j)}(x) \ll_j \Delta^{-1-j}$  holds for all  $j \in \mathbb{N}_0$ . Its Mellin transform

$$\widehat{\varrho}_\Delta(s) = \int_0^\infty \varrho_\Delta(x) x^{s-1} dx$$

is entire and satisfies

$$(2.2) \quad \frac{d^j}{ds^j} \widehat{\varrho}_\Delta(s) \ll_{\text{Re } s, A} (\Delta |s|)^{-A}$$

for all  $s \in \mathbb{C}$ , all  $j \in \mathbb{N}_0$  (in fact uniformly in  $j$ , although we will only apply it for fixed  $j$ ) and all integers  $A \in \mathbb{N}_0$ , as one confirms by integration by parts and differentiation under the integral sign. For  $x \in [0, \infty)$  we define

$$(2.3) \quad f_\Delta(x) = \int_0^\infty \varrho_\Delta(z) f_0\left(\frac{x}{z}\right) \frac{dz}{z} = \int_x^\infty \varrho_\Delta(z) \frac{dz}{z}.$$

Then, by (2.1) and (2.3),

$$(2.4) \quad 0 \leq f_\Delta(x) \leq 1, \quad f_\Delta = 1 \text{ on } [0, 1], \quad \text{supp}(f_\Delta) \subset [0, 1 + \Delta], \quad f_\Delta^{(j)}(x) \ll_j \Delta^{-j}$$

for  $x \in [0, \infty)$  and  $j \in \mathbb{N}_0$ . Further, we have

$$(2.5) \quad \widehat{f}_\Delta(s) = \widehat{\varrho}_\Delta(s)/s, \quad \widehat{f}_0(s) = 1/s.$$

We also note that  $\text{supp}(f'_\Delta) \subset [1, 1 + \Delta]$ . Thus,  $f_\Delta$  is indeed a smooth approximation to  $f_0$ , and as in [4, Lemma 24(i)] one shows that

$$(2.6) \quad \widehat{f}_\Delta(s) - \widehat{f}_0(s) \ll \min(\Delta, |s|^{-1}) \quad (1/100 \leq \text{Re } s \leq 2).$$

From (2.5) we now see that

$$(2.7) \quad \max(\widehat{f}_\Delta(s), \widehat{f}_0(s)) \ll |s|^{-1} \quad (1/100 \leq \text{Re } s \leq 2).$$

Let  $D$  be the differential operator given by  $(Df)(x) = xf'(x)$  for differentiable functions  $f$ . Then the Mellin transforms of  $Df$  and  $f$  (for, say, Schwartz class functions  $f$ ) are related by

$$(2.8) \quad \widehat{Df}(s) = s\widehat{f}(s).$$

Let  $X \geq 1$  be a parameter. We will also need a smooth approximation to the characteristic function of  $\frac{1}{2}X \leq |x| \leq X$ . To this end let  $1/10 \leq P \leq X/10$  be another parameter and let  $v$  be a non-negative smooth function with  $v(x) = 1$  for  $|x| \in [\frac{1}{2}X - P, X + P]$ ,  $v(x) = 0$  for  $|x| \notin [\frac{1}{2}X - 2P, X + 2P]$  and  $\|v^{(j)}\|_\infty \ll_j P^{-j}$  for all fixed  $j \in \mathbb{N}_0$ . We call such a function of *type*  $(X, P)$ . The Mellin transform  $\widehat{v}(s)$  of  $v$  is entire, and by partial integration one confirms easily the bound

$$(2.9) \quad \widehat{v}(s) \ll X^{\text{Re } s} (1 + |s|P/X)^{-2}$$

in fixed vertical strips.

**2.2. Certain sum transforms.** In this section we consider certain multiple sums with coprimality constraints on the variables of summation. Such sums occur in the counting process on the torsor, and we wish to remove the coprimality conditions by Möbius inversion.

Let  $\mathcal{B}$  denote the set of all  $(\mathbf{a}, \mathbf{d}, \mathbf{z}) \in \mathbb{Z}_0^3 \times \mathbb{Z}_0^3 \times \mathbb{Z}_0^3$  that satisfy the coprimality constraints

$$(2.10) \quad (a_1 z_1; a_2 z_2; a_3 z_3) = (d_i; d_j) = (z_i; z_j) = (d_k; z_k) = 1 \quad (1 \leq i < j \leq 3, 1 \leq k \leq 3),$$

Note that these conditions may also be written in the equivalent form

$$(2.11) \quad \begin{aligned} (d_i; d_j) &= (z_i; z_j) = (d_k; z_k) = 1 & (1 \leq i < j \leq 3, 1 \leq k \leq 3), \\ (a_1; a_2; a_3) &= (a_i; a_j; z_k) = 1 & (\{i, j, k\} = \{1, 2, 3\}). \end{aligned}$$

The significance of the set  $\mathcal{B}$  is that it appears naturally in the parametrization of the universal torsor (see Section 4).

**Lemma 2.1.** *Let  $G : \mathbb{Z}_0^{3 \times 3} \rightarrow \mathbb{C}$  be a function of compact support. Then*

$$(2.12) \quad \sum_{(\mathbf{a}, \mathbf{d}, \mathbf{z}) \in \mathcal{B}} G(\mathbf{a}, \mathbf{d}, \mathbf{z}) = \sum_{\mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g} \in \mathbb{N}^3} \sum_{h=1}^{\infty} \mu((\mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, h)) \sum_{\mathbf{a}, \mathbf{d}, \mathbf{z} \in \mathbb{Z}_0^3}^{\#} G(\mathbf{a}, \mathbf{d}, \mathbf{z}),$$

in which  $\sum^{\#}$  denotes that the sum is restricted to values  $\mathbf{a}, \mathbf{d}, \mathbf{z} \in \mathbb{Z}_0^3$  satisfying

$$(2.13) \quad [g_i; g_j; h] \mid a_k, \quad [b_i; b_j; f_k] \mid d_k, \quad [c_i; c_j; f_k; g_k] \mid z_k \quad (\{i, j, k\} = \{1, 2, 3\}).$$

*Proof.* Note that the simultaneous conditions (2.13) are equivalent to the divisibility conditions

$$(2.14) \quad b_k \mid (d_i; d_j), \quad c_k \mid (z_i; z_j), \quad f_k \mid (z_k; d_k), \quad g_k \mid (a_i; a_j; z_k), \quad h \mid (a_1; a_2; a_3) \quad (\{i, j, k\} = \{1, 2, 3\}).$$

Hence, on applying Möbius inversion to dissolve all 13 coprimality conditions in (2.11), one obtains the desired identity.  $\square$

In the sum on the right hand side of (2.12) it is often desirable to truncate all sums over  $b_j, c_j, f_j, g_j$  and  $h$  to an interval  $[1, T]$ , say. We wish to control the error in doing so, and for a discussion of this matter, some notation is in order. Suppose that the 13 variables  $b_j, c_j, f_j, g_j, h$  ( $1 \leq j \leq 3$ ) are labelled 1 to 13 in some fixed way, and let  $\mathcal{S}$  be a non-empty subset of  $\{1, 2, \dots, 13\}$ . If the label of some variable is in  $\mathcal{S}$ , then we say that the variable belongs to  $\mathcal{S}$ . If  $c_1$  belongs to  $\mathcal{S}$  then, by abuse of language, we write  $c_1 \in \mathcal{S}$ , and likewise for other variables.

We now claim that the inequality

$$\left| \sum_{\substack{\mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, h \\ x > T \text{ if } x \in \mathcal{S}}} \mu((\mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, h)) \sum_{\mathbf{a}, \mathbf{d}, \mathbf{z} \in \mathbb{Z}_0^3}^{\#} G(\mathbf{a}, \mathbf{d}, \mathbf{z}) \right| \leq \sum_{\substack{x \in \mathcal{S} \\ x > T}} \sum_{\mathbf{a}, \mathbf{d}, \mathbf{z}}^{\mathcal{S}} |G(\mathbf{a}, \mathbf{d}, \mathbf{z})|$$

holds, in which  $\sum^{\mathcal{S}}$  indicates that the sum is restricted to tuples  $(\mathbf{a}, \mathbf{d}, \mathbf{z})$  satisfying the divisibility conditions (2.14) for those variables that belong to  $\mathcal{S}$ . Note here that the outer sum consists of  $|\mathcal{S}|$  independent summations. For a proof of this inequality, we merely have to carry out all summations on the left hand side related to variables that do *not* belong to  $\mathcal{S}$ . Reversing the Möbius inversion formula, we then import one of the conditions (2.11) from each such sum. After this step, we are left with the summations over variables belonging to  $\mathcal{S}$ , we apply the triangle inequality and then drop the imported coprimality constraints to confirm the inequality as claimed above.

Equipped with this last inequality, we may indeed truncate all outer sums on the right hand side of (2.12). The inclusion-exclusion principle then allows us to conclude as follows.

**Lemma 2.2.** *Let  $T \geq 1$ . In the notation introduced in the preamble to this lemma, one has*

$$\sum_{(\mathbf{a}, \mathbf{d}, \mathbf{z}) \in \mathcal{B}} G(\mathbf{a}, \mathbf{d}, \mathbf{z}) = \sum_{|\mathbf{b}|, |\mathbf{c}|, |\mathbf{f}|, |\mathbf{g}|, h \leq T} \mu((\mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, h)) \sum_{\mathbf{a}, \mathbf{d}, \mathbf{z} \in \mathbb{Z}_0^3}^{\#} G(\mathbf{a}, \mathbf{d}, \mathbf{z}) + O\left(\sum_{\mathcal{S}} \sum_{\substack{x \in \mathcal{S} \\ x > T}} \sum_{\mathbf{a}, \mathbf{d}, \mathbf{z}}^{\mathcal{S}} |G(\mathbf{a}, \mathbf{d}, \mathbf{z})|\right).$$

The conditions (2.13) turn out to be significant in the future analysis, and we introduce the  $3 \times 3$ -tuple  $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^9$  with  $\alpha_j = (\alpha_{j1}, \alpha_{j2}, \alpha_{j3})$ , where whenever  $\{i, j, k\} = \{1, 2, 3\}$ , one takes

$$(2.15) \quad \alpha_{1k} = [g_i; g_j; h], \quad \alpha_{2k} = [b_i; b_j; f_k], \quad \alpha_{3k} = [c_i; c_j; f_k; g_k].$$

**2.3. An exponential sum.** In this section we examine an exponential sum of Kloosterman type. When  $a, b \in \mathbb{Z}$  and  $q \in \mathbb{N}$ , the classical Kloosterman sum is defined by

$$S(a, b; q) = \sum_{\substack{x=1 \\ (x; q)=1}}^q e\left(\frac{ax + b\bar{x}}{q}\right)$$

where, here and later, the bar denotes the multiplicative inverse with respect to a modulus that is always clear from the context; currently this modulus is  $q$ . Recall Weil's classical estimate  $|S(a, b; q)| \leq (a, b, q)^{1/2} \tau(q) q^{1/2}$ .

For  $r, x \in \mathbb{N}$ ,  $h, h_1, h_2 \in \mathbb{Z}$  we define the sum

$$(2.16) \quad S_{r,h}(h_1, h_2; x) = \sum_{\substack{\xi, \eta=1 \\ r\xi\eta \equiv -h \pmod{x}}}^x e\left(\frac{h_1\xi + h_2\eta}{x}\right).$$

We evaluate the sum (2.16) in terms of Kloosterman sums.

**Lemma 2.3.** *Let  $r, h, h_1, h_2, x$  be as above. Then one has  $S_{r,h}(h_1, h_2; x) = 0$  except when  $(r; x) \mid (h; h_1; h_2)$ , in which case  $S_{r,h}(h_1, h_2; x)$  equals*

$$\sum_{d(r; x) \mid (x; h; h_2)} d(r; x)^2 S\left(\frac{h_1}{(r; x)}, -\frac{h_2}{d(r; x)} \frac{\overline{r}}{(r; x)} \frac{h}{d(r; x)}, \frac{x}{d(r; x)}\right).$$

*Proof.* The sum (2.16) is empty unless  $(r; x) \mid h$ , which we assume from now on. We write  $r' = r/(r; x)$ ,  $h' = h/(r; x)$  and  $x' = x/(r; x)$ . Then

$$S_{r,h}(h_1, h_2; x) = \sum_{\substack{\xi, \eta=1 \\ \xi\eta \equiv -r'h' \pmod{x'}}}^x e\left(\frac{h_1\xi + h_2\eta}{x}\right) = \sum_{d \mid (x'; h')} \sum_{\substack{\xi=1 \\ (\xi; x'/d)=1}}^{x'/d} \sum_{\substack{\eta=1 \\ \eta \equiv -\overline{r'} \frac{h'}{d} \overline{\xi} \pmod{x'/d}}}^x e\left(\frac{h_1\xi d + h_2\eta}{x}\right).$$

The sum over  $\eta$  vanishes unless  $d(r; x) \mid h_2$ , and in the latter case we find that

$$S_{r,h}(h_1, h_2; x) = \sum_{d(r; x) \mid (x; h; h_2)} d(r; x) \sum_{\substack{\xi=1 \\ (\xi; x'/d)=1}}^{x/d} e\left(\frac{h_1\xi d - h_2 \overline{r'} \frac{h'}{d} \overline{\xi}}{x}\right).$$

The sum over  $\xi$  vanishes unless  $(r; x) \mid h_1$ , and we obtain the lemma.  $\square$

**Lemma 2.4.** *Let  $r, h, h_1, h_2, x$  be as in (2.16). Then  $S_{r,h}(0, 0; x) = 0$  unless  $(r; x) \mid h$ , in which case*

$$(2.17) \quad S_{r,h}(0, 0; x) = \sum_{d(r; x) \mid (x; h)} d(r; x)^2 \varphi\left(\frac{x}{(r; x)d}\right).$$

Further, when  $h_1 h_2 \neq 0$ , one has the inequalities

$$(2.18) \quad |S_{r,h}(0, h_2; x)| \leq \tau(h)(x; h h_2),$$

$$(2.19) \quad |S_{r,h}(h_1, 0; x)| \leq \tau(h)(x; h h_1),$$

$$(2.20) \quad |S_{r,h}(h_1, h_2; x)| \leq \tau^2(x)(r; x) x^{1/2} \left(\frac{h h_1}{(r; x)}; \frac{h h_2}{(r; x)}; x\right)^{1/2}.$$

*Proof.* The statements concerning  $S_{r,h}(0,0;x)$  are immediate from Lemma 2.3. By symmetry it is enough to prove one of the bounds (2.18) and (2.19), and we show the latter. Since  $h_1 \neq 0$ , the standard bound for Ramanujan sums  $|S(a,0;q)| \leq (a;q)$  suffices to conclude that

$$\begin{aligned} |S_{r,h}(h_1,0;x)| &\leq \sum_{d|(\frac{x}{(r;x)}, \frac{h}{(r;x)})} d(r;x)^2 \left( \frac{h_1}{(r;x)}; \frac{x}{d(r;x)} \right) \\ &\leq \tau(h) \left( \frac{x}{(r;x)}; \frac{h}{(r;x)} \right) (r;x)^2 \left( \frac{h_1}{(r;x)}; \frac{x}{(r;x)(\frac{x}{(r;x)}; \frac{h}{(r;x)})} \right) \\ &= \tau(h)(x; hh_1). \end{aligned}$$

Finally, for  $h_1 h_2 \neq 0$ , Weil's bound for Kloosterman sums yields

$$\begin{aligned} |S_{r,h}(h_1, h_2; x)| &\leq \tau(x) \sum_{d(x;r)|(x;h;h_2)} d^{1/2}(x;r) x^{1/2} \left( h_1; \frac{h_2 h}{d(r;x)}; \frac{x}{d} \right)^{1/2} \\ &\leq \tau^2(x)(x; h; h_2)^{1/2} (x;r)^{1/2} x^{1/2} \left( h_1; \frac{h_2 h}{(x; h; h_2)}; \frac{x(x;r)}{(x; h; h_2)} \right)^{1/2}, \end{aligned}$$

and (2.20) follows.  $\square$

## 2.4. Euler products.

**Lemma 2.5.** *Let  $a \in \mathbb{N}$  and  $X \geq 1$ ,  $1/10 \leq P \leq X/10$ . Further, let  $v$  be a function of type  $(X, P)$  as in Section 2.1. Then*

$$(2.21) \quad \sum_{n=1}^{\infty} \frac{\varphi(an)}{n^2} v(n) = \frac{\varphi(a)}{\zeta(2)} \prod_{p|a} \left( 1 - \frac{1}{p^2} \right)^{-1} \int_0^{\infty} v(x) \frac{dx}{x} + O\left(aX^{1/2}P^{-1} \log X\right).$$

*Proof.* Comparing Euler products, one easily confirms the formula

$$\sum_{n=1}^{\infty} \frac{\varphi(an)/\varphi(a)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} \prod_{p|a} \left( 1 - \frac{1}{p^s} \right)^{-1}$$

in  $\operatorname{Re} s > 2$ . By Mellin inversion we conclude that

$$\sum_{n \geq 1} \frac{\varphi(an)}{n^2} v(n) = \varphi(a) \int_{(1)} \frac{\zeta(s+1)}{\zeta(s+2)} \prod_{p|a} \left( 1 - \frac{1}{p^{s+2}} \right)^{-1} \widehat{v}(s) \frac{ds}{2\pi i}.$$

We shift the contour to  $\operatorname{Re} s = -1/2$ . The pole at  $s = 0$  contributes the main term on the right hand side of (2.21). Using (2.9) and Cauchy's inequality, we bound the remaining integral by

$$\ll \frac{a}{X^{1/2}} \int_{(-1/2)} \frac{|\zeta(s+1)|}{(1+|s|P/X)^2} |ds| \ll \frac{a}{P^{1/2}} \left( \int_{(-1/2)} \frac{|\zeta(s+1)|^2}{(1+|s|P/X)^2} |ds| \right)^{1/2}.$$

The standard bound  $\int_0^T |\zeta(1/2+it)|^2 dt \ll T \log T$  [26, Section 2.15] provides the bound  $(X/P) \log X/P$  for the last integral, which completes the proof.  $\square$

For  $\mathbf{r} = (r_1, r_2, r_3) \in \mathbb{N}^3$  and a prime  $p$  let  $\mathbf{r}(p) = (r_1^{v_p(r_1)}, r_2^{v_p(r_2)}, r_3^{v_p(r_3)})$ , where  $v_p$  is the usual  $p$ -adic valuation. With the shorthand notation  $r'_2 = r_2/(r_2; r_3)$ ,  $r'_3 = r_3/(r_2; r_3)$  we define

$$(2.22) \quad \mathcal{F}_{\mathbf{r}} = \frac{1}{\zeta(2)} \sum_{abc=r'_3} \frac{\mu(a)}{ab} \sum_{(d; r'_3/b)=1} \frac{(d; r'_2)(db; r_1)}{d^2} \sum_{fgh=\frac{db}{(db; r_1)}(r_2; r_3)} \frac{\mu(g)}{g} \prod_{p|(\frac{r'_2 a}{(d; r'_2)})} \left( 1 - \frac{1}{1+p} \right).$$



The function  $\mathbf{r} \mapsto \mathcal{F}_{\mathbf{r}}/\mathcal{F}_1$  is multiplicative in  $\mathbf{r}$ . We define the corresponding Euler factors

$$(2.23) \quad \mathcal{F}_1(p) = \left(1 - \frac{1}{p^2}\right) \sum_{\delta=0}^{\infty} \frac{1}{p^{2\delta}} \sum_{\substack{\varphi+\gamma \leq \delta \\ \gamma \leq 1}} \frac{(-1)^\gamma}{p^\gamma}, \quad \mathcal{F}_{\mathbf{r}}(p) = \mathcal{F}_1(p) \frac{\mathcal{F}_{\mathbf{r}}(p)}{\mathcal{F}_1},$$

so that  $\mathcal{F}_{\mathbf{r}} = \prod_p \mathcal{F}_{\mathbf{r}}(p)$ . Similarly, for the quantity  $\mathcal{E}_{\mathbf{r}}$  defined in (1.9), we define its Euler factors

$$\mathcal{E}_1(p) = \sum_{k=0}^{\infty} \frac{\varphi(p^k)}{p^{3k}}, \quad \mathcal{E}_{\mathbf{r}}(p) = \mathcal{E}_1(p) \frac{\mathcal{E}_{\mathbf{r}}(p)}{\mathcal{E}_1}.$$

With this notation, we have the following.

**Lemma 2.6.** a) Let  $d \in \mathbb{N}$  and  $\mathbf{r} \in \mathbb{N}^3$ . Then  $\mathcal{E}_{d\mathbf{r}} = d\mathcal{E}_{\mathbf{r}}$  and  $\mathcal{E}_{\mathbf{r}} \ll (r_1; r_2; r_3)(r_1 r_2 r_3)^\varepsilon$ .  
b) Let  $p$  be a prime and  $\mathbf{r} = (p^\alpha, p^\beta, 1)$  with  $0 \leq \beta \leq \alpha$ . Then  $\mathcal{F}_{\mathbf{r}}(p) = \mathcal{E}_{\mathbf{r}}(p)$ .

*Proof.* Put  $\alpha = v_p(r_1)$ ,  $\beta = v_p(r_2)$  and  $\gamma = v_p(r_3)$ . By symmetry, we may suppose that  $\alpha \geq \beta \geq \gamma$ , and then find that

$$(2.24) \quad \begin{aligned} \mathcal{E}_{\mathbf{r}}(p) &= 1 + \sum_{k=1}^{\infty} \frac{p^{k-1}(p-1)p^{\min(k,\alpha)+\min(k,\beta)+\min(k,\gamma)}}{p^{3k}} \\ &= \frac{p^{\gamma-\alpha-1}(p^\alpha(p+1)(1+\gamma-\beta+p(1-\gamma+\beta))-p^{\beta+1})}{p+1}. \end{aligned}$$

This formula shows on the one hand  $|\mathcal{E}_{\mathbf{r}}(p)| \leq p^\gamma(2+\beta)$ , on the other hand we see  $\mathcal{E}_{\mathbf{r}}(p) = d^{-1}\mathcal{E}_{d\mathbf{r}}(p)$  for  $d = p^\delta$  a power of  $p$ . Part (a) follows.

For (b), we note that

$$\begin{aligned} \mathcal{F}_{(p^\alpha, p^\beta, 1)} &= \left(1 - \frac{1}{p^2}\right) \sum_{d=0}^{\infty} \frac{p^{\min(d,\beta)+\min(d,\alpha)}}{p^{2d}} \sum_{\substack{f+g \leq \max(0, d-\alpha) \\ g \leq 1}} \frac{(-1)^g}{p^g} \sum_{k \leq \min(1, \max(0, \beta-d))} \frac{(-1)^k}{1+p^k} \\ &= \left(1 - \frac{1}{p^2}\right) \left( \sum_{d=0}^{\beta-1} \sum_{k=0}^1 \frac{(-1)^k}{1+p^k} + \sum_{d=\beta}^{\alpha-1} \frac{1}{p^{d-\beta}} + \sum_{d=\alpha}^{\infty} \frac{1}{p^{2d-\beta-\alpha}} \sum_{\substack{f+g \leq d-\alpha \\ g \leq 1}} \frac{(-1)^g}{p^g} \right) \\ &= \left(1 - \frac{1}{p^2}\right) \left( \frac{\beta p}{p+1} + \frac{p^{1-\alpha}(p^\alpha - p^\beta)}{p-1} + \frac{p^{1-\alpha+\beta}(1+p+p^2)}{(p-1)(p+1)^2} \right) = 1 + \beta + \frac{1-\beta}{p} - \frac{p^{\beta-\alpha}}{1+p}, \end{aligned}$$

which coincides with (2.24) if  $\gamma = 0$ .  $\square$

Our final lemma in this section investigates a multiple sum of multiplicative functions that comes up in the computation of the main term. We recall the definitions (2.15) and (1.3).

**Lemma 2.7.** In the range  $T \geq 1$  one has

$$(2.25) \quad \sum_{|\mathbf{b}|, |\mathbf{c}|, |\mathbf{f}|, |\mathbf{g}|, h \leq T} \mu((\mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, h)) \sum_{q \in \mathbb{N}} \frac{\varphi(q)}{q^3} \prod_{k=1}^3 \frac{(q; \alpha_{1k} \alpha_{2k})}{\alpha_{1k} \alpha_{2k} \alpha_{3k}} = C + O(T^{\varepsilon-1}).$$

*Proof.* The product  $C$  in (1.3) equals the completed sum on the left of (2.25), with all  $b_j, c_j, f_j, g_j$  and  $h$  running over all natural numbers. We first establish this claim.

The completed sum can be written as an Euler product where the Euler  $p$ -factor is given (formally) by the same sum, but with all variables of summation running over powers of  $p$ . The main observation is that there is no contribution from terms where  $p^2 \mid q$ . Indeed, for squarefree variables  $\mathbf{b}, \mathbf{f}, \mathbf{g}, h$ , the numbers  $\alpha_{1k}$  and  $\alpha_{2k}$  are squarefree, and hence,  $\alpha_{1k} \alpha_{2k}$  is cubefree. Then, whenever  $p^2 \mid q$ , we see that

$$v_p \left( \frac{(q; \alpha_{1k} \alpha_{2k})}{\alpha_{1k} \alpha_{2k} \alpha_{3k}} \right) = v_p \left( \frac{1}{\alpha_{3k}} \right)$$

is independent of  $v_p(h)$ , and consequently, the contribution from  $h = 1$  and  $h = p$  cancel out. Hence, we may introduce the multiplicative factor  $\mu(q)^2$  in the expression defining the completed sum. After this simplification, a mundane computation shows that the  $p$ -th Euler factor of this sum coincides with that of the product (1.3), as we have claimed.

It remains to estimate the error term introduced by completing the sum on the left. To this end we use Rankin's trick and bound the characteristic function on  $x \geq T$  by  $(x/T)^\xi$ , for some  $0 < \xi < 1$ . Thus it suffices to show that

$$(2.26) \quad \sum_{|\mathbf{b}|, |\mathbf{c}|, |\mathbf{f}|, |\mathbf{g}|, h} |\mu((\mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, h))| \sum_{q \in \mathbb{N}} \frac{\varphi(q)}{q^3} \prod_{k=1}^3 \frac{(q; \alpha_{1k} \alpha_{2k})}{\alpha_{1k} \alpha_{2k} \alpha_{3k}} \left( h^\xi + \sum_{j=1}^3 (b_j^\xi + c_j^\xi + f_j^\xi + g_j^\xi) \right)$$

is absolutely convergent. To see this, first note that the rightmost factor in the preceding display is a sum of 13 summands, and it is then sufficient to show absolute convergence with only one of these summands present. Irrespective of which summand is present, we are reduced to multiple sum of multiplicative terms that we may formally rewrite as an Euler product. As before, its  $p$ -th Euler factor arises from letting all variables run through powers of  $p$ . Again as before, since  $\alpha_{1k} \alpha_{2k}$  is cubefree, it is clear that terms affecting convergence in the Euler  $p$ -factor come from  $q \mid p^2$ . Another mundane computation then shows that the  $p$ -th Euler factor under consideration is of the form  $1 + O(p^{\xi-2})$ . We take  $\xi = 1 - \varepsilon$  to ensure absolute convergence of the Euler product. This completes the proof.  $\square$

**2.5. Mellin inversion formulae.** Our first lemma in this section expresses the Fourier integral (1.10) as a Mellin integral. This features the meromorphic function

$$(2.27) \quad K(s) = \frac{\Gamma(s) \cos(\pi s/2) (1 - 2^{s-1})^2}{(1-s)^2}.$$

**Lemma 2.8.** *Let  $\mathbf{r} \in \mathbb{N}^3$  and  $X_1, X_2, X_3, Y_1, Y_2, Y_3 \geq 1$ . Then, whenever the positive numbers  $c_1, c_2$  satisfy  $c_1 + c_2 < 1$ , one has*

$$\mathcal{I}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) = \frac{64}{\pi} \int_{(c_2)} \int_{(c_1)} \frac{(X_1 Y_1)^{1-s} (X_2 Y_2)^{1-t} (X_3 Y_3)^{s+t}}{r_1^s r_2^t r_3^{1-s-t}} K(s) K(t) K(1-s-t) \frac{ds dt}{(2\pi i)^2}.$$

In particular, choosing  $c_1 = c_2 = 1/3$ , we see that

$$(2.28) \quad \mathcal{I}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) \ll \frac{(X_1 X_2 X_3 Y_1 Y_2 Y_3)^{2/3}}{(r_1 r_2 r_3)^{1/3}}.$$

*Proof.* Let  $B(X, Y)$  denote the region  $\frac{1}{2}X \leq |x| \leq X$ ,  $\frac{1}{2}Y \leq |y| \leq Y$ . For  $\alpha, r \in \mathbb{R}$ , one has

$$(2.29) \quad \int_{B(X, Y)} e(\alpha r x y) d(x, y) = \frac{2(\text{Si}(\frac{1}{2}\pi\alpha r X Y) - 2\text{Si}(\pi\alpha r X Y) + \text{Si}(2\pi\alpha r X Y))}{\pi\alpha r},$$

where

$$\text{Si}(x) = \int_0^x \frac{\sin(t)}{t} dt$$

is the integral sine. By [12, 6.246.1, 8.230.1], the identity

$$(2.30) \quad \int_0^\infty \frac{2(\text{Si}(x/4) - 2\text{Si}(x/2) + \text{Si}(x))}{x/2} x^{s-1} dx = 4 \int_{(c)} K(s) \frac{ds}{2\pi i}$$

holds for  $0 < c < 1$ , and hence, for the same  $c$ , Mellin inversion yields

$$\int_{B(X, Y)} e(\alpha r x y) d(x, y) = 4 \int_{(c)} K(s) (2\pi r |\alpha|)^{-s} (XY)^{1-s} \frac{ds}{2\pi i}.$$

We use this formula twice for the integration over  $x_1, y_1, x_2, y_2$  with contours  $(c_1), (c_2)$  such that  $c_1, c_2 > 0$ ,  $c_1 + c_2 < 1$ . Then we integrate over  $x_3, y_3$  using (2.29) and finally integrate over  $\alpha$  by (2.30). This gives the desired formula.  $\square$

The following lemma computes explicitly a certain multiple Mellin integral whose integrand is a rational function.

**Lemma 2.9.** *Fix  $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$  with  $\operatorname{Re} z_1 = \operatorname{Re} z_2 = 1/3$ , and for  $\mathbf{y} = (y_6, y_7, y_8, y_9) \in \mathbb{C}^4$  let*

$$F_{\mathbf{z}}(\mathbf{y}) = y_6 y_7 \left( \frac{1 - z_1}{2} - y_6 - y_7 \right) 2y_8 2y_9 \left( \frac{1 - z_2}{2} - 2y_8 - 2y_9 \right) (1 - z_1 - z_2 - y_6 - 2y_8) \\ \times (z_2 - y_7 - 2y_9) \left( \frac{3z_1 + z_2 - 2}{2} + y_6 + y_7 + 2y_8 + 2y_9 \right).$$

Then

$$\frac{1}{(2\pi i)^4} \int_{(\frac{1}{15})} \int_{(\frac{1}{15})} \int_{(\frac{1}{15})} \int_{(\frac{1}{15})} (F_{\mathbf{z}}(\mathbf{y}))^{-1} dy_6 dy_7 dy_8 dy_9 = \frac{2}{(1 - z_1)(1 - z_2)z_1 z_2 (z_1 + z_2)(1 - z_1 - z_2)}.$$

*Proof.* This can be obtained by straightforward contour shifts. We shift all contours successively to the far left (the opposite direction would be possible, too). Each time we pick up two poles, and the remaining integral vanishes in the limit. If we first compute the innermost integral over  $y_6$  in this way and then divide by  $2\pi i$ , we arrive at

$$\frac{4(4y_9 - z_2 + 1)}{(2y_7 + z_1 - 1)(2y_8 + z_1 + z_2 - 1)(2y_7 + 4y_9 + z_1 - z_2)(4y_8 + 4y_9 + 2z_1 + z_2 - 1)} \\ \times \left[ y_7 2y_8 2y_9 \left( \frac{1 - z_2}{2} - 2y_8 - 2y_9 \right) (z_2 - y_7 - 2y_9) \right]^{-1}.$$

We integrate this over  $y_7$  and again divide by  $2\pi i$ , then obtaining

$$\frac{4y_9 - z_2 + 1}{(2y_8 + z_1 + z_2 - 1)(4y_8 + 4y_9 + 2z_1 + z_2 - 1)y_8 y_9 \left( \frac{1 - z_2}{2} - 2y_8 - 2y_9 \right)} \\ \times \frac{1 + z_2}{(1 - z_1)(z_1 + z_2)(4y_9 - z_2 + 1)(2y_9 - z_2)}.$$

Again, integrating this over  $y_8$  and dividing by  $2\pi i$ , one gets the function

$$\frac{1 + z_2}{(1 - z_1)(z_1 + z_2)y_9(2y_9 - z_2)} \cdot \frac{2(z_2 - 1)}{z_1(1 - z_1 - z_2)(4y_9 + z_2 - 1)(z_2 - 1 - 4y_9)}.$$

Finally,

$$\int (\dots) \frac{dy_9}{2\pi i} = \frac{(1 + z_2)2(z_2 - 1)}{(1 - z_1)(z_1 + z_2)z_1(1 - z_1 - z_2)} \cdot \frac{-1}{(z_2 - 1)^2(1 + z_2)z_2},$$

and the claim follows.  $\square$

The double Mellin integral in the next lemma is related to the archimedean density of our algebraic variety, cf. Lemma 8.4.

**Lemma 2.10.** *We have*

$$\int_{(\frac{1}{3})} \int_{(\frac{1}{3})} \frac{\Gamma(z_1)\Gamma(z_2)\Gamma(1 - z_1 - z_2) \cos(\frac{\pi z_1}{2}) \cos(\frac{\pi z_2}{2}) \cos(\frac{\pi(1 - z_1 - z_2)}{2})}{(1 - z_1)(1 - z_2)z_1 z_2 (z_1 + z_2)(1 - z_1 - z_2)} \frac{dz_1 dz_2}{(2\pi i)^2} = \frac{\pi}{8}(\pi^2 - 3 + 24 \log 2).$$

*Proof.* We call the left hand side  $\mathcal{I}$ . First we note the Mellin formula

$$\int_0^\infty \left( \int_y^\infty \frac{\sin t}{t^2} dt \right) y^{u-1} dy = \int_0^\infty \left( \int_y^\infty \frac{\cos t}{t} dt + \frac{\sin y}{y} \right) y^{u-1} dy = \frac{\Gamma(u) \cos(\pi u/2)}{u(1 - u)}$$

that holds for  $0 < \operatorname{Re} u < 1$  ([12, 6.246.2] and [12, 3.761.4]). Hence, by Mellin inversion, we have

$$\mathcal{I} = \int_0^\infty \left( \int_y^\infty \frac{\sin t}{t^2} dt \right)^3 dy = \int_0^\infty \left( \int_1^\infty \frac{\sin yt}{t^2} dt \right)^3 \frac{dy}{y^3}.$$

We apply the Fubini-Tonelli theorem and see that we may exchange the order of integrations. This yields the formula

$$\mathcal{I} = \int_{[1,\infty]^3} \frac{T(\mathbf{t})}{t_1^2 t_2^2 t_3^2} d\mathbf{t}$$

where

$$T(\mathbf{t}) = \int_0^\infty \frac{\sin yt_1 \sin yt_2 \sin yt_3}{y^3} dy.$$

Let  $\operatorname{sgn} \beta$  denote the sign of the real number  $\beta$ . Then, on writing the sin-function in terms of exponentials, a standard application of the residue theorem shows that

$$T(\mathbf{t}) = \frac{\pi}{16} \left( (t_1 + t_2 + t_3)^2 - (t_1 + t_2 - t_3)^2 \operatorname{sgn}(t_1 + t_2 - t_3) \right. \\ \left. - (t_1 - t_2 + t_3)^2 \operatorname{sgn}(t_1 - t_2 + t_3) - (-t_1 + t_2 + t_3)^2 \operatorname{sgn}(-t_1 + t_2 + t_3) \right).$$

If  $t_1 > 0$ ,  $t_2 > 0$  and  $|t_1 - t_2| \geq 1$ , a straightforward computation shows (split at  $|t_1 - t_2|$  and  $t_1 + t_2$ )

$$\int_1^\infty \frac{T(\mathbf{t})}{t_1^2 t_2^2 t_3^2} dt_3 = \frac{\pi}{16t_1^2 t_2^2} \left( 8 \min(t_1, t_2) \log |t_1 - t_2| - 8 \frac{t_1 t_2}{t_1 + t_2} + 4(t_1 + t_2) \log \frac{t_1 + t_2}{|t_1 - t_2|} + 8 \frac{t_1 t_2}{t_1 + t_2} \right) \\ = \pi \frac{(t_1 + t_2) \log(t_1 + t_2) - |t_1 - t_2| \log |t_1 - t_2|}{4t_1^2 t_2^2},$$

while for  $t_1 > 0$ ,  $t_2 > 0$  and  $|t_1 - t_2| < 1$ , a slightly simpler computation gives

$$\int_1^\infty \frac{T(\mathbf{t})}{t_1^2 t_2^2 t_3^2} dt_3 = \frac{\pi}{16t_1^2 t_2^2} \left( 4(t_1 + t_2) \log(t_1 + t_2) - \frac{2(t_1 + t_2 - 1)(t_1 + t_2 + (t_1 - t_2)^2)}{t_1 + t_2} + 8 \frac{t_1 t_2}{t_1 + t_2} \right) \\ = \frac{\pi(1 - (t_1 - t_2)^2 + 2(t_1 + t_2) \log(t_1 + t_2))}{8t_1^2 t_2^2}.$$

Let

$$\mathcal{T}_1 = \{(t_1, t_2) \in (0, \infty)^2 : |t_1 - t_2| \geq 1\}, \quad \mathcal{T}_2 = \{(t_1, t_2) \in (0, \infty)^2 : |t_1 - t_2| < 1\}.$$

We then have a natural decomposition  $\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2$ , where

$$\mathcal{I}_1 = \int_{\mathcal{T}_1} \pi \frac{(t_1 + t_2) \log(t_1 + t_2) - |t_1 - t_2| \log |t_1 - t_2|}{4t_1^2 t_2^2} d\mathbf{t}, \\ \mathcal{I}_2 = \int_{\mathcal{T}_2} \frac{\pi(1 - (t_1 - t_2)^2 + 2(t_1 + t_2) \log(t_1 + t_2))}{8t_1^2 t_2^2} d\mathbf{t}.$$

Obvious substitutions deliver that

$$\mathcal{I}_1 = 2\pi \int_1^\infty \int_1^\infty \frac{(r + 2t) \log(r + 2t) - r \log r}{4(r + t)^2 t^2} dt dr = 2\pi \int_0^1 \int_1^\infty \frac{(1 + 2rt) \log(1 + 2rt) - 2rt \log r}{4t^2 r(1 + rt)^2} dt dr,$$

and a similar computation produces

$$\mathcal{I}_2 = 2\pi \int_0^1 \int_1^\infty \frac{1 - r^2 + 2(r + 2t) \log(r + 2t)}{8(r + t)^2 t^2} dt dr.$$

The  $t$ -integrals in the final expression for  $\mathcal{I}_j$  have an elementary primitive, and a tedious computation yields

$$\mathcal{I} = \frac{\pi}{4} \int_0^1 f(r) dr$$

where  $f$  is defined by

$$-8r^3(1 + r)f(r) = 8r \log 2 + r(1 + r)(-2 + r + r^2 + 8r \log 2) + 4r^3(1 + r) \log(1 + 1/r) \log r \\ + 4r^4 \log(4r) + 2(1 + r(3 + r + r^2 + 2r^3)) \log(1 + r) - 2r(2 + r)^2 \log(2 + r) \\ - 2r^2(1 + 2r)^2 \log(1 + 2r).$$

Now let

$$\mathrm{Li}_2(z) = - \int_0^z \frac{\log(1-t)}{t} dt = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

be the Dilogarithm, see [16] for basic and more advanced properties of this function. By brute force one then checks that a primitive of  $f$  is given by  $F$ , where

$$\begin{aligned} 8F(r) = & -\frac{1}{r} - r + \frac{8 \log 2}{r} - 8r \log 2 + 4 \log \left(1 + \frac{1}{r}\right) + 4r \log \left(1 + \frac{1}{r}\right) + 4 \log r - 4r \log r \\ & - 2 \log r \log \left(1 + \frac{1}{r}\right) - 4r \log r \log \left(1 + \frac{1}{r}\right) - 2(\log r)^2 - \log(1+r) + \frac{\log(1+r)}{r^2} \\ & + \frac{4 \log(1+r)}{r} - 4r \log(1+r) + 2 \log r \log(1+r) - 4 \log(2+r) - \frac{8 \log(2+r)}{r} \\ & + 4 \log(1+2r) - 2 \log 2 \log(1+2r) + 8r \log(1+2r) - 2 \log(1+r) \log(1+2r) \\ & - 2 \mathrm{Li}_2(-1-r) - 2 \mathrm{Li}_2(-r) - 2 \mathrm{Li}_2(-2r) - 2 \mathrm{Li}_2(-1-2r). \end{aligned}$$

In order to evaluate  $F(1) - F(0)$ , we need to evaluate  $\mathrm{Li}_2(-3) + 2 \mathrm{Li}_2(-2) - \mathrm{Li}_2(-1)$ . It is well-known that  $\mathrm{Li}_2(-1) = -\pi^2/12$ . Moreover, one confirms by differentiation that the function

$$x \mapsto 2 \mathrm{Li}_2(x) + \mathrm{Li}_2(1-x^2) - 2 \mathrm{Li}_2(-1/x) + 2 \log(1-x^2) \log x - (\log x)^2$$

is constant, and it takes the value  $\pi^2/6$ , as can be seen by substituting  $x = 1$ . For  $x = -2$  we use  $\mathrm{Li}_2(-1/2) = \frac{1}{12}\pi^2 - \frac{1}{2}(\log 2)^2$  [16, (1.16)] to conclude that

$$2 \mathrm{Li}_2(-2) + \mathrm{Li}_2(-3) = -\frac{\pi^2}{3} - 2 \log 2 \log 3.$$

Altogether, this gives  $\mathcal{I} = \frac{\pi}{4}(F(1) - F(0)) = \frac{\pi}{8}(\pi^2 - 3 + 24 \log 2)$  as required.  $\square$

### 3. AN ASYMPTOTIC FORMULA

This section is devoted to a proof of Proposition 1.2. By symmetry we can assume that

$$(3.1) \quad r_1 X_1 Y_1 \leq r_2 X_2 Y_2 \leq r_3 X_3 Y_3, \quad X_2 \leq Y_2.$$

To begin with, we make the two additional assumptions

$$(3.2) \quad (r_1; r_2; r_3) = 1, \quad r_2 X_2 X_3 \asymp r_3 X_3 Y_3 = Z,$$

say.

We start our argument by smoothing the summation conditions: let  $P_1, P_2, P_3, Q_1, Q_2, Q_3$  satisfy  $1/10 \leq P_j \leq X_j/10$ ,  $1/10 \leq Q_j \leq Y_j/10$ , and for  $1 \leq j \leq 3$  let  $v_j$  be a function of type  $(X_j, P_j)$  and  $w_j$  be a function of type  $(Y_j, Q_j)$ , cf. Section 2.1. Let

$$N_{\mathbf{r}}^{(1)}(\mathbf{X}, \mathbf{Y}) = \sum' v_1(x_1) v_2(x_2) v_3(x_3) w_1(y_1) w_2(y_2) w_3(y_3),$$

where the prime indicates summation over  $(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}_0^6$  satisfying (1.8). We choose

$$P_j = \frac{X_j}{\Xi^\delta}, \quad Q_j = \frac{Y_j}{\Xi^\delta}, \quad \Xi = \min(X_1, X_2, X_3, Y_1, Y_2, Y_3)$$

for some  $0 < \delta < 1$  to be specified later. A simple divisor argument shows

$$(3.3) \quad N_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) = N_{\mathbf{r}}^{(1)}(\mathbf{X}, \mathbf{Y}) + O(X_1 Y_1 Z^{1+\varepsilon} \Xi^{-\delta}).$$

Since  $(r_1; r_2; r_3) = 1$ , we may write

$$(3.4) \quad N_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) = \sum_{(r_2; r_3) | x_1 y_1} v_1(x_1) w_1(y_1) M\left(\frac{r_2}{(r_2, r_3)}, \frac{r_3}{(r_2, r_3)}, \frac{r_1 x_1 y_1}{(r_2, r_3)}\right)$$

where

$$M(r'_2, r'_3, h) = \sum_{h+r'_2x_2y_2+r'_3x_3y_3=0} v_2(x_2)v_3(x_3)w_2(y_2)w_3(y_3).$$

We now manipulate  $M(r'_2, r'_3, h)$  for  $h \neq 0$ ,  $(r'_2; r'_3) = 1$ . We have

$$\begin{aligned} M(r'_2, r'_3, h) &= \sum_{x_2} v_2(x_2) \sum_{r_3x_3y_3 \equiv -h \pmod{r'_2|x_2|}} v_3(x_3)w_2\left(\frac{-r'_3x_3y_3-h}{r'_2x_2}\right)w_3(y_3) \\ &= \sum_{x_2} v_2(x_2) \sum_{\substack{\xi, \eta \pmod{r'_2|x_2|} \\ r'_3\xi\eta \equiv -h \pmod{r'_2|x_2|}}} \sum_{\substack{x_3 \equiv \xi \pmod{r'_2|x_2|} \\ y_3 \equiv \eta \pmod{r'_2|x_2|}}} W_{r'_2, r'_3, h}(x_3, y_3; x_2), \end{aligned}$$

where

$$W_{r'_2, r'_3, h}(x_3, y_3; x_2) = v_3(x_3)w_2\left(\frac{-r'_3x_3y_3-h}{r'_2x_2}\right)w_3(y_3).$$

By the Poisson summation formula we obtain

$$(3.5) \quad M(r'_2, r'_3, h) = \sum_{x_2} \frac{v_2(x_2)}{(r'_2x_2)^2} \sum_{h_1, h_2 \in \mathbb{Z}} \mathcal{W}_{r'_2, r'_3, h}\left(\frac{h_1}{r'_2|x_2|}, \frac{h_2}{r'_2|x_2|}; x_2\right) S_{r'_3, h}(h_1, h_2; r'_2|x_2|),$$

where the exponential sum  $S_{r'_3, h}(h_1, h_2; r'_2|x_2|)$  was defined in (2.16), and where

$$\mathcal{W}_{r'_2, r'_3, h}(\xi, \eta; x_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{r'_2, r'_3, h}(x_3, y_3; x_2) e(-x_3\xi - y_3\eta) dx_3 dy_3$$

is the Fourier transform with respect to the first two variables. By partial integration it is easy to see that

$$\mathcal{W}_{r'_2, r'_3, h}(\xi, \eta; x_2) \ll_{A, B} X_3 Y_3 \left( \left( \frac{1}{P_3} + \frac{r'_3 Y_3}{r'_2 Q_2 X_2} \right) \frac{1}{|\xi|} \right)^A \left( \left( \frac{1}{Q_3} + \frac{r'_3 X_3}{r'_2 Q_2 X_2} \right) \frac{1}{|\eta|} \right)^B$$

holds for any  $A, B \geq 0$  and  $x_2 \asymp X_2$ . By (2.20), the contribution of the terms  $h_1 h_2 \neq 0$  to (3.5) is therefore at most

$$(3.6) \quad \ll Z^\varepsilon(r'_2; h)^{1/2} (r'_2)^{1/2} X_3 Y_3 X_2^{3/2} \left( \frac{1}{P_3} + \frac{r'_3 Y_3}{r'_2 Q_2 X_2} \right) \left( \frac{1}{Q_3} + \frac{r'_3 X_3}{r'_2 Q_2 X_2} \right).$$

Similarly, by (2.18) and (2.19) the contribution of the terms  $h_1 h_2 = 0$ ,  $(h_1, h_2) \neq (0, 0)$  is at most

$$(3.7) \quad \ll Z^\varepsilon \frac{(r'_2; h)}{r'_2} X_3 Y_3 \left( \frac{1}{P_3} + \frac{1}{Q_3} + \frac{r'_3(X_3 + Y_3)}{r'_2 Q_2 X_2} \right).$$

Moreover, by (2.17), the contribution of the central term equals

$$\sum_{(r'_3; x_2) | h} \frac{v_2(x_2)}{(r'_2 x_2)^2} \mathcal{W}_{r'_2, r'_3, h}(0, 0; x_2) \sum_{d(r'_3; x_2) | (r'_2 x_2; h)} d(r'_3; x_2)^2 \varphi\left(\frac{r'_2 x_2}{(r'_3; x_2)d}\right).$$

We substitute this back into (3.4) and sum the error terms (3.6) and (3.7) over  $x_1 \asymp X_1$  and  $y_1 \asymp Y_1$  (recall (3.1)). We continue to write  $r'_2 = r_2/(r_2; r_3)$  and  $r'_3 = r_3/(r_2; r_3)$  and see in this way that  $N_{\mathbf{r}}^{(1)}(\mathbf{X}, \mathbf{Y})$  equals the expression

$$\begin{aligned} N_{\mathbf{r}}^{(2)}(\mathbf{X}, \mathbf{Y}) &= \sum_{(r_2; r_3) | x_1 y_1} v_1(x_1)w_1(y_1) \sum_{(r'_3; x_2) | \frac{r_1 x_1 y_1}{(r_2; r_3)}} \frac{v_2(x_2)}{(r'_2 x_2)^2} \sum_{d(r'_3; x_2) | (r'_2 x_2; \frac{r_1 x_1 y_1}{(r_2; r_3)})} d(r'_3; x_2)^2 \\ &\quad \times \varphi\left(\frac{r'_2 x_2}{(r'_3; x_2)d}\right) \int_{\mathbb{R}} \int_{\mathbb{R}} v_3(x_3)w_2\left(\frac{-r_3 x_3 y_3 - r_1 x_1 y_1}{r_2 x_2}\right)w_3(y_3) dx_3 dy_3, \end{aligned}$$

up to an error that does not exceed

$$\begin{aligned}
(3.8) \quad & \ll Z^\varepsilon \frac{(r'_2; r_1) X_1 Y_1 X_3 Y_3}{r'_2} \left( \frac{1}{P_3} + \frac{1}{Q_3} + \frac{r'_3(X_3 + Y_3)}{r'_2 Q_2 X_2} \right) \\
& + Z^\varepsilon X_1 Y_1 X_3 Y_3 (r'_2; r_1)^{1/2} (r'_2)^{1/2} X_2^{3/2} \left( \frac{1}{P_3} + \frac{r'_3 Y_3}{r'_2 Q_2 X_2} \right) \left( \frac{1}{Q_3} + \frac{r'_3 X_3}{r'_2 Q_2 X_2} \right) \\
& \ll X_1 Y_1 Z^{1+\varepsilon} (\Xi^{\delta-1} + \Xi^{2\delta-\frac{1}{2}}).
\end{aligned}$$

In the sum defining  $N_{\mathbf{r}}^{(2)}(\mathbf{X}, \mathbf{Y})$ , we pull the  $d$ -sum outside and introduce a new variable  $b = (x_2; r'_3)$ . Then the summation conditions for  $x_1, y_1, x_2$  become

$$\frac{db}{(db; r_1)}(r_2; r_3) \mid x_1 y_1, \quad (x_2; r'_3) = b, \quad \frac{db}{(db; r'_2)} \mid x_2.$$

Writing  $x_2 = x'_2 db / (d; r'_2)$ , the last two conditions are equivalent to  $(x'_2 d; r'_3/b) = 1$ . Hence we can rewrite the main term as

$$\begin{aligned}
N_{\mathbf{r}}^{(2)}(\mathbf{X}, \mathbf{Y}) &= \sum_{b \mid r'_3} \sum_{(d; r'_3/b)=1} \frac{(d; r'_2)^2}{(r'_2)^2 d} \sum_{\frac{db}{(db; r_1)}(r_2; r_3) \mid x_1 y_1} \sum_{(x_2; r'_3/b)=1} \varphi\left(\frac{r'_2 x_2}{(d; r'_2)}\right) \frac{v_2(db x_2 / (d; r'_2))}{x_2^2} \\
&\quad \times v_1(x_1) w_1(y_1) \int_{\mathbb{R}} \int_{\mathbb{R}} v_3(x_3) w_2\left(\frac{-r_3 x_3 y_3 - r_1 x_1 y_1}{r_2 db x_2 / (d; r'_2)}\right) w_3(y_3) dx_3 dy_3.
\end{aligned}$$

By Möbius inversion this equals

$$\begin{aligned}
& \sum_{abc=r'_3} \frac{\mu(a)}{a^2} \sum_{(d; r'_3/b)=1} \frac{(d; r'_2)^2}{(r'_2)^2 d} \sum_{fgh=\frac{db(r_2; r_3)}{(db; r_1)}} \mu(g) \sum_{x_1, y_1} \sum_{x_2} \varphi\left(\frac{r'_2 a x_2}{(d; r'_2)}\right) \frac{v_2(dba x_2 / (d; r'_2))}{x_2^2} \\
& \quad \times v_1(fg x_1) w_1(hg y_1) \int_{\mathbb{R}} \int_{\mathbb{R}} v_3(x_3) w_2\left(\frac{-r_3 x_3 y_3 - r_1 f g^2 h x_1 y_1}{r_2 dba x_2 / (d; r'_2)}\right) w_3(y_3) dx_3 dy_3.
\end{aligned}$$

We execute the  $x_2$ -sum by Lemma 2.5. This introduces an error not exceeding

$$\begin{aligned}
(3.9) \quad & \ll Z^\varepsilon \sum_{abc=r'_3} \frac{1}{a^2} \sum_d \frac{(d; r'_2)^2}{(r'_2)^2 d} \sum_{fgh=\frac{db(r_2; r_3)}{(db; r_1)}} \frac{X_1 Y_1}{f g^2 h} X_3 Y_3 \frac{r'_2 a x_2}{(d; r'_2)} \left( \frac{X_2(d; r'_2)}{dba} \right)^{\frac{1}{2}} \left( \frac{dba}{(d; r'_2) P_2} + \frac{dba Y_2}{(d; r'_2) Q_2 X_2} \right) \\
& \ll Z^\varepsilon \frac{(r'_3; r_1)^{1/2}}{r'_2} X_1 Y_1 X_3 Y_3 X_2^{1/2} \left( \frac{1}{P_2} + \frac{Y_2}{Q_2 X_2} \right) \ll X_1 Y_1 Z^{1+\varepsilon} \Xi^{\delta-\frac{1}{2}}.
\end{aligned}$$

Next we execute the  $x_1$ -sum by Poisson summation and keep only the central term. This introduces an error no larger than

$$\begin{aligned}
(3.10) \quad & \ll \sum_{abc=r'_3} \frac{1}{a^2} \sum_d \frac{(d; r'_2)^2}{(r'_2)^2 d} \sum_{fgh=\frac{db(r_2; r_3)}{(db; r_1)}} \frac{r'_2 a}{(d; r'_2)} \frac{Y_1}{gh} X_3 Y_3 \left( \frac{X_1}{P_1} + \frac{r_1 Y_1 X_1}{r_2 X_2 Q_2} \right) \\
& \ll Z^\varepsilon \frac{Y_1 X_3 Y_3}{r'_2} \left( \frac{X_1}{P_1} + \frac{r_1 Y_1 X_1}{r_2 X_2 Q_2} \right) \ll X_1 Y_1 Z^{1+\varepsilon} \Xi^{\delta-1};
\end{aligned}$$

here we applied (3.1). Finally we execute the  $y_1$ -sum by Poisson summation and keep only the central term. This introduces an error of

$$(3.11) \quad \ll Z^\varepsilon \frac{X_1 X_3 Y_3}{r'_2} \left( \frac{Y_1}{Q_1} + \frac{r_1 Y_1 X_1}{r_2 X_2 Q_2} \right) \ll X_1 Y_1 Z^{1+\varepsilon} \Xi^{\delta-1}.$$

Hence, up to an error described by (3.9) – (3.11), we can write  $N_{\mathbf{r}}^{(2)}(\mathbf{X}, \mathbf{Y})$  in the form

$$\begin{aligned} N_{\mathbf{r}}^{(3)}(\mathbf{X}, \mathbf{Y}) &= \sum_{abc=r'_3} \frac{\mu(a)}{a^2} \sum_{(d;r'_3/b)=1} \frac{(d;r'_2)^2}{(r'_2)^2 d} \sum_{fgh=\frac{db}{(db;r_1)}(r_2;r_3)} \mu(g) \frac{\varphi(r'_2 a/(d;r'_2))}{\zeta(2)} \\ &\quad \times \prod_{p|\frac{r'_2 a}{(d;r'_2)}} \left(1 - \frac{1}{p^2}\right)^{-1} \int_{\mathbb{R}^5} v_2\left(\frac{dbax_2}{(d;r'_2)}\right) v_1(fgx_1) w_1(hgy_1) v_3(x_3) \\ &\quad \times w_2\left(\frac{-r_3 x_3 y_3 - r_1 f g^2 h x_1 y_1}{r_2 db a x_2/(d;r'_2)}\right) w_3(y_3) |x_2|^{-1} d(x_1, x_2, x_3, y_1, y_3). \end{aligned}$$

A change of variables yields

$$N_{\mathbf{r}}^{(3)}(\mathbf{X}, \mathbf{Y}) = \mathcal{F}_{\mathbf{r}} \mathcal{J}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}),$$

where

$$\begin{aligned} \mathcal{F}_{\mathbf{r}} &= r_2 \sum_{abc=r'_3} \frac{\mu(a)}{a^2} \sum_{(d;r'_3/b)=1} \frac{(d;r'_2)^2}{(r'_2)^2 d} \sum_{fgh=\frac{db}{(db;r_1)}(r_2;r_3)} \frac{\mu(g)}{f g^2 h} \frac{\varphi(r'_2 a/(d;r'_2))}{\zeta(2)} \prod_{p|\frac{r'_2 a}{(d;r'_2)}} \left(1 - \frac{1}{p^2}\right)^{-1} \\ &= \frac{1}{\zeta(2)} \sum_{abc=r'_3} \frac{\mu(a)}{ab} \sum_{(d;r'_3/b)=1} \frac{(d;r'_2)(db;r_1)}{d^2} \sum_{fgh=\frac{db}{(db;r_1)}(r_2;r_3)} \frac{\mu(g)}{g} \prod_{p|\frac{r'_2 a}{(d;r'_2)}} \left(1 - \frac{1}{1+p}\right) \end{aligned}$$

is seen to coincide with the definition (2.22), and where

$$\mathcal{J}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) = \frac{1}{r_2} \int_{\mathbb{R}^5} v_2(x_2) v_1(x_1) w_1(y_1) v_3(x_3) w_2\left(\frac{-r_3 x_3 y_3 - r_1 x_1 y_1}{r_2 x_2}\right) w_3(y_3) |x_2|^{-1} d(\mathbf{x}, y_1, y_3).$$

Note that  $\mathcal{F}_{\mathbf{r}} \ll (r_1 r_2 r_3)^\varepsilon$ . Turning to  $\mathcal{J}$ , we apply Fourier inversion to see that

$$\begin{aligned} w_2\left(\frac{-r_3 x_3 y_3 - r_1 x_1 y_1}{r_2 x_2}\right) &= \int_{\mathbb{R}} \int_{\mathbb{R}} w_2(y_2) e(y_2 \alpha) dy_2 e\left(\alpha \frac{r_3 x_3 y_3 + r_1 x_1 y_1}{r_2 x_2}\right) d\alpha \\ &= r_2 |x_2| \int_{\mathbb{R}} \int_{\mathbb{R}} w_2(y_2) e(\alpha(r_1 x_1 y_1 + r_2 x_2 y_2 + r_3 x_3 y_3)) dy_2 d\alpha. \end{aligned}$$

This double integral is not absolutely convergent, but the integral over  $\alpha$  is absolutely convergent, and this is all we need to justify the following interchange of integrals:

$$\mathcal{J}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) = \int_{\mathbb{R}} \int_{\mathbb{R}^6} v_1(x_1) w_1(y_1) v_2(x_2) w_2(y_2) v_3(x_3) w_3(y_3) e(\alpha(r_1 x_1 y_1 + r_2 x_2 y_2 + r_3 x_3 y_3)) d(\mathbf{x}, \mathbf{y}) d\alpha.$$

Finally we remove the smooth weight functions. To this end we observe that the estimate

$$(3.12) \quad \int_{X/2}^X \int_{Y/2}^Y e(\alpha r x y) dx dy \ll \min\left(XY, \frac{1}{r|\alpha|}\right)$$

(cf. (2.29)) holds uniformly for  $r, X, Y \geq 1$ ,  $\alpha \in \mathbb{R}$ , and one also has

$$\int_{B(X,Y)} e(\alpha r x y) d(x, y) - \int_{\mathbb{R}^2} v(x) w(y) e(\alpha r x y) d(x, y) \ll \min\left(PY + QX, \frac{1}{r|\alpha|}\right),$$

where, as before,  $B(X, Y)$  denotes the region  $\frac{1}{2}X \leq |x| \leq X$ ,  $\frac{1}{2}Y \leq |y| \leq Y$ . This shows that

$$(3.13) \quad \mathcal{J}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) = \mathcal{I}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) + O\left(X_1 Y_1 Z^{1+\varepsilon} \Xi^{-\delta}\right),$$

with  $\mathcal{I}_{\mathbf{r}}$  as in (1.10). Collecting the error terms (3.3), (3.8), (3.9), (3.10), (3.11) and (3.13) and choosing  $\delta = 1/6$ , we have now proved the asymptotic relation

$$(3.14) \quad N_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) = \mathcal{F}_{\mathbf{r}} \mathcal{I}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) + O\left(\frac{X_1 Y_1 (r_2 X_2 Y_2)^{1+\varepsilon}}{\min(X_1, X_2, X_3, Y_1, Y_2, Y_3)^{1/6}}\right),$$



yet subject to the additional assumptions (3.2).

For fixed  $\mathbf{r}$  and  $X_1 = X_2 = X_3 = Y_1 = Y_2 = Y_3 = W$ , it follows easily from Lemma 2.8 that  $\mathcal{I}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) \asymp_{\mathbf{r}} W^4$ , while the error term is  $O_{\mathbf{r}}(W^{23/6+\varepsilon})$ . Moreover, both  $N_{\mathbf{r}}(\mathbf{X}, \mathbf{Y})$  and  $\mathcal{I}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y})$  are symmetric in  $r_1, r_2, r_3$ . Thus letting  $W \rightarrow \infty$ , we conclude that  $\mathcal{F}_{\mathbf{r}}$  is symmetric in  $r_1, r_2, r_3$ , provided  $(r_1; r_2; r_3) = 1$ . By (2.23), also the Euler factors  $\mathcal{F}_{\mathbf{r}}(p)$  are symmetric, for all primes  $p$ . (This can be checked directly, too, but requires some computation.) By Lemma 2.6b we now infer that  $\mathcal{E}_{\mathbf{r}} = \mathcal{F}_{\mathbf{r}}$  if  $(r_1; r_2; r_3) = 1$ , and we have proved Proposition 1.2 under the assumptions (3.2).

It remains to remove these extra assumptions. First, should it be the case that

$$(3.15) \quad 10(r_1 X_1 Y_1 + r_2 X_2 Y_2) \leq r_3 X_3 Y_3,$$

then clearly  $N_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) = 0$ . We proceed to show that (3.15) also implies that  $\mathcal{J}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) = 0$ . Indeed, formally integrating by parts in the  $\alpha$ -integral, we obtain that

$$\begin{aligned} \mathcal{J}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) &= \int_{\mathbb{R}} \int_{\mathbb{R}^6} v_1(x_1) w_1(y_1) v_2(x_2) w_2(y_2) v_3(x_3) w_3(y_3) \left( -\frac{r_1 x_1 y_1 + r_2 x_2 y_2}{r_3 x_3 y_3} \right)^n \\ &\quad \times e(\alpha(r_1 x_1 y_1 + r_2 x_2 y_2 + r_3 x_3 y_3)) d(\mathbf{x}, \mathbf{y}) d\alpha \end{aligned}$$

for any positive integer  $n$ . In particular, we conclude that  $\mathcal{J}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}) = 0$  whenever (3.15) holds. To justify this formal manipulation, we observe that (by partial integration in any of the  $x$  or  $y$ -variables) the  $\alpha$ -integral is rapidly decaying at  $\pm\infty$ . Hence we can truncate it (smoothly) with an arbitrarily small error, pull it inside and integrate by parts, pull it outside and complete the range of integration again with an arbitrarily small error. This argument shows that the proposition holds trivially under the assumption (3.15), and hence we can drop our initial assumption  $r_2 X_2 Y_2 \asymp r_3 X_3 Y_3$ .

By (3.12) we see that the  $\alpha$ -integral in the definition of  $\mathcal{I}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y})$  is absolutely convergent, hence we can make a change of variables to conclude that

$$\mathcal{I}_{d\mathbf{r}}(\mathbf{X}, \mathbf{Y}) = d\mathcal{I}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y})$$

holds for all  $d \in \mathbb{N}$ . Together with Lemma 2.6a we see

$$\mathcal{E}_{d\mathbf{r}} \mathcal{I}_{d\mathbf{r}}(\mathbf{X}, \mathbf{Y}) = \mathcal{E}_{\mathbf{r}} \mathcal{I}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y})$$

But  $N_{d\mathbf{r}}(\mathbf{X}, \mathbf{Y}) = N_{\mathbf{r}}(\mathbf{X}, \mathbf{Y})$ , whence we may dismiss the assumption that  $(r_1; r_2; r_3) = 1$ . The proof of the proposition is complete.

#### 4. THE ELEMENTARY PART OF THE ARGUMENT

**4.1. The universal torsor.** We keep the notation introduced in Section 2.2. Let  $\mathcal{A}$  denote the set of all  $(\mathbf{a}, \mathbf{d}, \mathbf{z}) \in \mathbb{Z}_0^3 \times \mathbb{N}^3 \times \mathbb{Z}_0^3$  that satisfy the lattice equation

$$(4.1) \quad a_1 d_1 + a_2 d_2 + a_3 d_3 = 0$$

and the coprimality constraints (2.10) or equivalently (2.11). We recall that the four six-tuples  $(\pm \mathbf{x}, \pm \mathbf{y})$  satisfying (1.1) are representatives of the same point on  $V^\circ$ . The following result from [2, Section 2] provides a parametrization of the points on  $V^\circ$ .

**Lemma 4.1.** *The mapping  $\mathcal{A} \rightarrow V^\circ$  defined by*

$$(4.2) \quad \begin{aligned} x_1 &= a_1 z_1, & x_2 &= a_2 z_2, & x_3 &= a_3 z_3, \\ y_1 &= d_2 d_3 z_1, & y_2 &= d_1 d_3 z_2, & y_3 &= d_1 d_2 z_3 \end{aligned}$$

is 4-to-1.

**4.2. Upper bounds.** We will use frequently the following lattice point count [14, Lemma 3].

**Lemma 4.2.** *Let  $\mathbf{v} \in \mathbb{Z}^3$  be primitive and let  $H_i > 0$  ( $1 \leq i \leq 3$ ). Then the number of primitive  $\mathbf{u} \in \mathbb{Z}^3$  that satisfy  $u_1v_1 + u_2v_2 + u_3v_3 = 0$  and that lie in the box  $|u_i| \leq H_i$  ( $1 \leq i \leq 3$ ), is  $O(1 + H_1H_2|v_3|^{-1})$ .*

We introduce the following notation. For  $\mathbf{X} = (X_1, X_2, X_3)$ ,  $\mathbf{Y} = (Y_1, Y_2, Y_3)$  with  $X_j, Y_j \geq 1$ ,  $H \geq 1$  and  $\mathbf{r}, \boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\zeta} \in \mathbb{N}^3$  let  $\mathcal{V}_{\mathbf{r};(\boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\zeta})}(\mathbf{X}, \mathbf{Y}, H)$  be the set of 9-tuples  $(\mathbf{a}, \mathbf{d}, \mathbf{z}) \in \mathbb{Z}_0^9$  satisfying

$$(4.3) \quad |a_j z_j| \leq X_j \quad (1 \leq j \leq 3), \quad |d_i d_j z_k| \leq Y_k \quad (\{i, j, k\} = \{1, 2, 3\}),$$

$$(4.4) \quad \min(|a_1|, |a_2|, |a_3|, |d_1|, |d_2|, |d_3|, |z_1|, |z_2|, |z_3|) \leq H,$$

$$(4.4) \quad r_1 a_1 d_1 + r_2 a_2 d_2 + r_3 a_3 d_3 = 0,$$

$$(4.5) \quad \alpha_j \mid a_j, \quad \delta_j \mid d_j, \quad \zeta_j \mid z_j \quad (1 \leq j \leq 3).$$

**Lemma 4.3.** *Let  $H, X_j, Y_j, \mathbf{r}, \boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\zeta}$  be as in the preceding paragraph, and write  $Z = |\mathbf{X}|_1 + |\mathbf{Y}|_1$ . Then*

$$(4.6) \quad |\mathcal{V}_{\mathbf{r};(\boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\zeta})}(\mathbf{X}, \mathbf{Y}, H)| \ll \tau^2 \left( \prod_{j=1}^3 r_j \alpha_j \delta_j \right) \frac{(X_1 X_2 X_3)^{2/3} (Y_1 Y_2 Y_3)^{1/3}}{(\alpha_1 \alpha_2 \alpha_3 \delta_1 \delta_2 \delta_3)^{2/3} \zeta_1 \zeta_2 \zeta_3} (\log Z)^2 \log H.$$

*Proof.* We use some ideas from [2, Section 7]. Changing variables

$$r_k \mapsto r_k \alpha_k \delta_k, \quad X_k \mapsto \frac{X_k}{\alpha_k \zeta_k}, \quad Y_k \mapsto \frac{Y_k}{\delta_k \zeta_k}$$

with  $\{i, j, k\} = \{1, 2, 3\}$ , the general version of (4.6) is reduced to the case where  $\boldsymbol{\alpha} = \boldsymbol{\delta} = \boldsymbol{\zeta} = (1, 1, 1)$ , so that we may concentrate on the latter from now on. Accordingly, we drop  $\boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\zeta}$  from the notation as these are now fixed to  $(1, 1, 1)$ . Without loss of generality we may also assume that  $(r_1; r_2; r_3) = 1$ .

We first consider the restricted set  $\tilde{\mathcal{V}}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}, H)$  of  $(\mathbf{a}, \mathbf{d}, \mathbf{z}) \in \mathcal{V}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}, H)$  satisfying the additional condition

$$(4.7) \quad (r_1 d_1; r_2 d_2; r_3 d_3) = (r_1 a_1; r_2 a_2; r_3 a_3) = 1.$$

We cut the  $a_j$  and  $d_j$  in dyadic ranges  $A_j < a_j \leq 2A_j$  and  $D_j < d_j \leq 2D_j$ . Lemma 4.2 shows that the number of  $(\mathbf{a}, \mathbf{d}) \in \mathbb{N}^6$  satisfying (4.7) and (4.1) in a given dyadic range is at most

$$(4.8) \quad \ll \min(D_1 D_2 D_3, A_1 A_2 A_3) + \frac{\prod_j (A_j D_j)}{\max_j (A_j D_j)} \ll \prod_{j=1}^3 (A_j D_j)^{2/3}.$$

Summing this over  $\mathbf{z} = (z_1, z_2, z_3)$  with  $|z_j| \leq Z_j$  for  $1 \leq j \leq 3$ , we obtain that for each 6-tuple of dyadic ranges  $A_j, D_j$  the contribution is

$$(4.9) \quad \ll \min\left(\frac{X_1}{A_1}, \frac{Y_1}{D_1 D_2}, Z_1\right) \min\left(\frac{X_2}{A_2}, \frac{Y_2}{D_1 D_3}, Z_2\right) \min\left(\frac{X_3}{A_3}, \frac{Y_3}{D_1 D_2}, Z_3\right) \prod_{j=1}^3 (A_j D_j)^{2/3}.$$

If we define  $E_j = D_1 D_2 D_3 / D_j$ , the above simplifies to

$$(4.10) \quad \ll \prod_{j=1}^3 A_j^{2/3} E_j^{1/3} \min\left(\frac{X_j}{A_j}, \frac{Y_j}{E_j}, Z_j\right).$$

Notice now that as  $(\nu_1, \nu_2, \nu_3)$  runs over  $\mathbb{N}^3$ , the triples  $(\nu_2 + \nu_3, \nu_1 + \nu_3, \nu_1 + \nu_2)$  take each value in  $\mathbb{N}^3$  at most once. Hence we can replace a summation in which the  $D_j = 2^{\nu_j}$  run over powers of 2 by a sum in which the  $E_j$  run over powers of 2. It remains to sum (4.10) over  $A_j$  and  $E_j$  which run over powers of 2. For any  $X, Y, H \geq 1$  we have

$$(4.11) \quad \sum_{A=2^\nu} A^{2/3} E^{1/3} \min\left(\frac{X}{A}, \frac{Y}{E}, H\right) \ll X^{2/3} \min(Y, HE)^{1/3}$$

uniformly in  $1 \leq E \leq Y$ , and

$$(4.12) \quad \sum_{E=2^\nu} A^{2/3} E^{1/3} \min\left(\frac{X}{A}, \frac{Y}{E}\right) \ll X^{2/3} Y^{1/3}$$

uniformly in  $1 \leq A \leq X$ .

If  $|a_j| \leq H$  for some  $1 \leq j \leq 3$ , then summing (4.10) first over  $E_1, E_2, E_3$  using (4.12) and then trivially over  $A_1, A_2, A_3$ , we arrive at

$$|\tilde{\mathcal{V}}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}, H)| \ll (X_1 X_2 X_3)^{2/3} (Y_1 Y_2 Y_3)^{1/3} (\log Z)^2 \log H.$$

If  $|z_j| \leq H$  for some  $1 \leq j \leq 3$ , then summing (4.10) first over  $A_1, A_2, A_3$  using (4.11) and then trivially over  $E_1, E_2, E_3$ , we arrive again at

$$|\tilde{\mathcal{V}}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}, H)| \ll (X_1 X_2 X_3)^{2/3} (Y_1 Y_2 Y_3)^{1/3} (\log Z)^2 \log H.$$

Finally, if  $|d_j| \leq H$  for some  $1 \leq j \leq 3$ , then again we sum (4.10) first over  $A_1, A_2, A_3$  using (4.11). Noticing that

$$\frac{E_i E_k}{H^2} \ll E_j \ll E_i E_k, \quad \{i, j, k\} = \{1, 2, 3\},$$

there are at most  $(\log Z)^2 \log H$  terms in the sum over  $E_1, E_2, E_3$ , and again we obtain

$$(4.13) \quad |\tilde{\mathcal{V}}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}, H)| \ll (X_1 X_2 X_3)^{2/3} (Y_1 Y_2 Y_3)^{1/3} (\log Z)^2 \log H.$$

With the above bound for  $|\tilde{\mathcal{V}}_{\mathbf{r}}(\mathbf{X}, \mathbf{Y}, H)|$  we can easily finish the proof. If  $(r_1 a_1; r_2 a_2; r_3 a_3) = a$  and  $(r_1 d_1; r_2 d_2; r_3 d_3) = d$ , we now apply our bounds with  $X_j(a; r_1 r_2 r_3)/a$  in place of  $X_j$  and  $Y_k(d; r_1 r_2 r_3)^2/d^2$  in place of  $Y_k$ . Summing over  $a$  and  $d$  yields (4.6) in all cases.  $\square$

For  $B, H \geq 1$  and  $\mathbf{r}, \boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\zeta} \in \mathbb{N}^3$  let  $\mathcal{V}_{\mathbf{r};(\boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\zeta})}(B, H)$  be the set of 9-tuples  $(\mathbf{a}, \mathbf{d}, \mathbf{z}) \in \mathbb{Z}_0^9$  satisfying

$$(4.14) \quad \max_{1 \leq j \leq 3} (|a_j z_j|)^2 \max_{\{i, j, k\} = \{1, 2, 3\}} (|d_i d_j z_k|) \leq B$$

as well as (4.3), (4.4) and (4.5).

Summing (4.6) over  $O(\log B)$  tuples  $(\mathbf{X}, \mathbf{Y}) = (4^j, 4^j, 4^j, 4^{2-2j}B, 4^{2-2j}B, 4^{2-2j}B)$ , we may now conclude as follows.

**Lemma 4.4.** *For  $B, H \geq 1$  and  $\mathbf{r}, \boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\zeta} \in \mathbb{N}^3$  we have*

$$|\mathcal{V}_{\mathbf{r};(\boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\zeta})}(B, H)| \ll \tau^2 \left( \prod_{j=1}^3 r_j \alpha_j \delta_j \right) \frac{B}{(\alpha_1 \alpha_2 \alpha_3 \delta_1 \delta_2 \delta_3)^{2/3} \zeta_1 \zeta_2 \zeta_3} (\log B)^3 \log H.$$

A continuous version is given by the following lemma.

**Lemma 4.5.** *Let  $B, H \geq 1$  and let  $\mathcal{S} = \mathcal{S}(B, H)$  denote the set of points  $(\mathbf{a}, \mathbf{d}, \mathbf{z}) \in [1, \infty)^9$  satisfying (4.3) and (4.14). Then*

$$\int_{\mathcal{S}} \frac{1}{(a_1 a_2 a_3 d_1 d_2 d_3)^{1/3}} d(\mathbf{a}, \mathbf{d}, \mathbf{z}) \ll B (\log B)^3 (\log H).$$

*Proof.* This is a simpler version of the proof of Lemmas 4.3 and 4.4, so we can be brief. We cut the variables into ranges  $A_j \leq a_j \leq 2A_j$ ,  $D_j \leq d_j \leq 2D_j$  and  $z_j \leq Z_j$ . Fix  $1 \leq X, Y, \leq B$  and consider first the contribution of points where  $a_j z_j \leq X$  for  $1 \leq j \leq 3$  and  $d_i d_j z_k \leq Y$  for  $\{i, j, k\} = \{1, 2, 3\}$ . Then the integral restricted to this set is

$$\ll \min\left(\frac{X}{A_1}, \frac{Y}{D_2 D_3}, Z_1\right) \min\left(\frac{X}{A_2}, \frac{Y}{D_1 D_3}, Z_2\right) \min\left(\frac{X}{A_3}, \frac{Y}{D_1 D_2}, Z_3\right) \prod_{j=1}^3 (A_j D_j)^{2/3}$$

as in (4.9). Arguing as in the proof of Lemma 4.3 with  $X_1 = X_2 = X_3 = X$ ,  $Y_1 = Y_2 = Y_3 = Y$ , we see that total contribution of all choices  $A_j, D_j, Z_j$  is  $\ll X^2 Y (\log B)^2 \log H$ , as in (4.13). Finally summing over  $O(\log B)$  tuples  $(\mathbf{X}, \mathbf{Y})$ , we complete the proof.  $\square$

## 5. THE ANALYTIC PART OF THE ARGUMENT

**5.1. Preliminary transformations.** We begin with some notation. In an effort to establish a sufficiently compact presentation we write a typical vector  $\mathbf{x} \in \mathbb{C}^9$  as

$$(5.1) \quad \mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = (x_{11}, x_{12}, x_{13}; x_{21}, x_{22}, x_{23}; x_{31}, x_{32}, x_{33}).$$

For a typical index we write  $\ell = (i, j) \in \{1, 2, 3\} \times \{1, 2, 3\}$ . In the notation of the previous sections we write  $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = (\mathbf{a}, \mathbf{d}, \mathbf{z})$ . For vectors  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $\mathbf{y} = (y_1, \dots, y_n)$  we write

$$\mathbf{x} \cdot \mathbf{y} = (x_1 y_1, \dots, x_n y_n).$$

With coordinates on  $\mathbb{Z}_0^9$  given by (5.1), let  $\chi : \mathbb{Z}_0^9 \rightarrow [0, 1]$  be the characteristic function on the set defined by  $x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23} = 0$ , and let  $\psi : \mathbb{Z}_0^9 \rightarrow [0, 1]$  be the characteristic function on the set of 9-tuples satisfying the coprimality conditions corresponding to (2.11), that is,

$$\begin{aligned} (x_{2i}; x_{2j}) &= (x_{3i}; x_{3j}) = (x_{2k}; x_{3k}) = 1 & (1 \leq i < j \leq 3, 1 \leq k \leq 3), \\ (x_{11}; x_{22}; x_{33}) &= (x_{1i}; x_{1j}; x_{3k}) = 1 & (\{i, j, k\} = \{1, 2, 3\}). \end{aligned}$$

For  $0 \leq \Delta < 1$ , we then put

$$(5.2) \quad F_{\Delta, B}(\mathbf{x}) = \prod_{l=1}^3 \prod_{\substack{1 \leq i < j \leq 3 \\ \{i, j, k\} = \{1, 2, 3\}}} f_{\Delta} \left( \frac{|(x_{1l}x_{3l})^2 x_{2i}x_{2j}x_{3k}|}{B} \right)$$

where  $f_{\Delta}$  was defined in (2.3). Finally, we introduce the sum

$$N_{\Delta}(B) = \frac{1}{4} \sum_{\mathbf{x}_1 \in \mathbb{Z}_0^3} \sum_{\mathbf{x}_2 \in \mathbb{N}^3} \sum_{\mathbf{x}_3 \in \mathbb{Z}_0^3} \chi(\mathbf{x}) \psi(\mathbf{x}) F_{\Delta, B}(\mathbf{x}).$$

We extend the summation over  $\mathbf{x}_2$  to  $\mathbb{Z}_0^3$  and include an additional factor  $1/8$ . This does not change the value of  $N_{\Delta}(B)$ , but it is notationally slightly more convenient. Recalling the height condition (1.2), it follows from Lemma 4.1 and (4.2) that  $N_0(B) = N(B)$ , but it is analytically easier to treat the smooth version  $N_{\Delta}(B)$  for  $\Delta > 0$ . But an asymptotic formula of  $N_{\Delta}(B)$  with  $\Delta > 0$  is all what we require because from (2.4) we readily see that the chain of inequalities

$$(5.3) \quad N_{\Delta}(B(1 - \Delta)) \leq N(B) \leq N_{\Delta}(B)$$

holds. We remove the function  $\psi$ , which captures the coprimality conditions, by Lemma 2.2 and estimate the error term by Lemma 4.4 with  $H = B$  and  $\mathbf{r} = (1, 1, 1)$ . For  $T \geq 1$  this gives

$$(5.4) \quad N_{\Delta}(B) = N_{\Delta, T}(B) + O \left( B(\log B)^4 T^{\varepsilon - 1/3} \right),$$

where

$$(5.5) \quad N_{\Delta, T}(B) = \frac{1}{32} \sum_{|\mathbf{b}|, |\mathbf{c}|, |\mathbf{f}|, |\mathbf{g}|, h \leq T} \mu((\mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, h)) \sum_{\mathbf{x}_1 \in \mathbb{Z}_0^3} \sum_{\mathbf{x}_2 \in \mathbb{Z}_0^3} \sum_{\mathbf{x}_3 \in \mathbb{Z}_0^3} \chi(\boldsymbol{\alpha} \cdot \mathbf{x}) F_{\Delta, B}(\boldsymbol{\alpha} \cdot \mathbf{x}),$$

and  $\boldsymbol{\alpha}$  is as in (2.15). The factor  $T^{\varepsilon - 1/3}$  in the error term of (5.4) comes from observing that for every subset  $\mathcal{S}$  in the error term of Lemma 2.2, the corresponding variables  $x \in \mathcal{S}$  occur by Lemma 4.4 at least with an exponent  $4/3 - \varepsilon$  in the denominator.

From now on, the analysis will frequently feature multiple Mellin-Barnes integrals over specific vertical lines, and we write  $\int_{(\beta)}^{(n)}$  for an  $n$ -fold iterated such integral; the lines of integration will be clear from the context or otherwise specified in the text. If all  $n$  integrations are over the same line  $(\beta)$ , then we write this as  $\int_{(\beta)}^{(n)}$ .

We continue to manipulate  $N_{\Delta, T}(B)$ . Let  $\Delta > 0$ , and recall the definition (5.2). We then use Mellin inversion and the notation (1.13) to recast  $N_{\Delta, T}(B)$  as

$$(5.6) \quad \frac{1}{32} \sum_{|\mathbf{b}|, |\mathbf{c}|, |\mathbf{f}|, |\mathbf{g}|, h \leq T} \mu((\mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, h)) \int_{(1)}^{(9)} \sum_{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{Z}_0^3} \frac{\chi(\boldsymbol{\alpha} \cdot \mathbf{x})}{\boldsymbol{\alpha}^{\mathbf{v}} \mathbf{x}^{\mathbf{v}}} \prod_{\ell} \left( \widehat{f}_{\Delta}(s_{\ell}) B^{s_{\ell}} \right) \frac{ds}{(2\pi i)^9},$$

where  $\mathbf{v} = \mathbf{v}(\mathbf{s}) = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3$  is defined by

$$\begin{aligned}
 v_{11} &= 2(s_{11} + s_{12} + s_{13}), & v_{12} &= 2(s_{21} + s_{22} + s_{23}), & v_{13} &= 2(s_{31} + s_{32} + s_{33}), \\
 v_{21} &= s_{11} + s_{12} + s_{21} + s_{22} + s_{31} + s_{32}, \\
 v_{22} &= s_{11} + s_{13} + s_{21} + s_{23} + s_{31} + s_{33}, \\
 (5.7) \quad v_{23} &= s_{12} + s_{13} + s_{22} + s_{23} + s_{32} + s_{33}, \\
 v_{31} &= 2(s_{11} + s_{12} + s_{13}) + s_{13} + s_{23} + s_{33}, \\
 v_{32} &= 2(s_{21} + s_{22} + s_{23}) + s_{12} + s_{22} + s_{32}, \\
 v_{33} &= 2(s_{31} + s_{32} + s_{33}) + s_{11} + s_{21} + s_{31},
 \end{aligned}$$

and  $\ell$  runs over  $\{1, 2, 3\}^2$ . In view of (2.2) and (2.5), the  $\mathbf{s}$ -integral in (5.6) is absolutely convergent.

At this point it would be possible to evaluate the  $\mathbf{x}_3$ -sum directly in terms of Riemann's zeta function. This is because  $\chi(\boldsymbol{\alpha} \cdot \mathbf{x})$  is independent of  $\mathbf{x}_3$ . However, it is easier to treat  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$  on equal footing. By partial summation and then unfolding the integral, we have

$$\begin{aligned}
 \sum_{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{Z}_0^3} \frac{\chi(\boldsymbol{\alpha} \cdot \mathbf{x})}{\boldsymbol{\alpha}^{\mathbf{v}} \mathbf{x}^{\mathbf{v}}} &= \frac{1}{\boldsymbol{\alpha}^{\mathbf{v}}} \left( \prod_{\ell} v_{\ell} \right) \int_{[1, \infty)^9} \sum_{0 < |x_{\ell}| \leq X_{\ell}} \chi(\boldsymbol{\alpha} \cdot \mathbf{x}) \mathbf{X}^{-\mathbf{v}-1} d\mathbf{X} \\
 &= \frac{1}{\boldsymbol{\alpha}^{\mathbf{v}}} \left( \prod_{\ell} \frac{v_{\ell}}{1 - 2^{-v_{\ell}}} \right) \int_{[1, \infty)^9} \sum_{\frac{1}{2} X_{\ell} < |x_{\ell}| \leq X_{\ell}} \chi(\boldsymbol{\alpha} \cdot \mathbf{x}) \mathbf{X}^{-\mathbf{v}-1} d\mathbf{X}.
 \end{aligned}$$

In the notation of Proposition 1.2 this equals

$$\frac{1}{\boldsymbol{\alpha}^{\mathbf{v}}} \left( \prod_{\ell} \frac{v_{\ell}}{1 - 2^{-v_{\ell}}} \right) \int_{[1, \infty)^9} N_{\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2}(\mathbf{X}_1, \mathbf{X}_2) \cdot 8 \prod_{j=1}^3 \left( [X_{3j}] - \left\lfloor \frac{X_{3j}}{2} \right\rfloor \right) \mathbf{X}^{-\mathbf{v}-1} d\mathbf{X}.$$

We would like to evaluate this integral with the aid of Proposition 1.2, and this is successful if we replace the region  $[1, \infty)^9$  with

$$(5.8) \quad \mathcal{R}_{\delta} := \{\mathbf{x} \in [1, \infty) : \min(x_1, \dots, x_n) \geq \max(x_1, \dots, x_n)^{\delta}\}$$

for  $0 < \delta < 1/10$ , say. With this in mind, for such  $\delta$ , we define

$$\begin{aligned}
 (5.9) \quad N_{\Delta, T, \delta}(B) &= \frac{1}{4} \sum_{|\mathbf{b}|, |\mathbf{c}|, |\mathbf{f}|, |\mathbf{g}|, h \leq T} \mu((\mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, h)) \int_{(1)}^{(9)} \frac{1}{\boldsymbol{\alpha}^{\mathbf{v}}} \prod_{\ell} \frac{v_{\ell}}{1 - 2^{-v_{\ell}}} \\
 &\quad \times \int_{\mathcal{R}_{\delta}} N_{\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2}(\mathbf{X}_1, \mathbf{X}_2) \prod_{j=1}^3 \left( [X_{3j}] - \left\lfloor \frac{X_{3j}}{2} \right\rfloor \right) \mathbf{X}^{-\mathbf{v}-1} d\mathbf{X} \prod_{\ell} \widehat{f}_{\Delta}(s_{\ell}) B^{s_{\ell}} \frac{ds}{(2\pi i)^9}.
 \end{aligned}$$

The next lemma estimates the error that we infer by throwing away the information in the cusps.

**Lemma 5.1.** *Uniformly for  $B \geq 1$ ,  $T \geq 1$ ,  $0 < \Delta < 1$ ,  $0 < \delta < 1/10$ , one has*

$$N_{\Delta, T, \delta}(B) = N_{\Delta, T}(B) + O(T^{13} \delta B (\log B)^4).$$

We postpone the proof to the end of this section. We will eventually choose  $T$  to be a small power of  $\log B$  and  $\delta, \Delta$  small powers of  $(\log B)^{-1}$ , see (5.28).

**5.2. The error term.** We are now ready to insert the asymptotic formula from Proposition 1.2, and we also insert the obvious asymptotic formula

$$[X] - \left\lfloor \frac{X}{2} \right\rfloor = \frac{X}{2} + O(1)$$

along with the trivial bound

$$N_{\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2}(\mathbf{X}_1, \mathbf{X}_2) \ll \frac{(X_{11} X_{12} X_{13} X_{21} X_{22} X_{23})^{1+\varepsilon}}{\max(X_{11} X_{21}, X_{12} X_{22}, X_{13} X_{23})},$$

which follows from a simple divisor argument. This gives

$$(5.10) \quad N_{\alpha_1, \alpha_2}(\mathbf{X}_1, \mathbf{X}_2) \prod_{j=1}^3 \left( [X_{3j}] - \left\lfloor \frac{X_{3j}}{2} \right\rfloor \right) = \mathcal{E}_{\alpha_1, \alpha_2} \mathcal{I}_{\alpha_1, \alpha_2}(\mathbf{X}_1, \mathbf{X}_2) \frac{X_{31} X_{32} X_{33}}{8} + \Psi_{\alpha_1, \alpha_2}(\mathbf{X}),$$

where in the case when  $\mathbf{X} \in \mathcal{R}_\delta$ , one has the estimate

$$(5.11) \quad \begin{aligned} \Psi_{\alpha_1, \alpha_2}(\mathbf{X}) &\ll \frac{X_{31} X_{32} X_{33} \prod_{i=1}^2 \prod_{j=1}^3 (\alpha_{ij} X_{ij})^{1+\varepsilon}}{\max(X_{11} X_{21}, X_{12} X_{22}, X_{13} X_{23}) \min_\ell (X_\ell^{1/6})} \\ &\leq \left( \prod_{i=1}^2 \prod_{j=1}^3 \alpha_{ij}^{1+\varepsilon} \right) \left( \prod_{i=1}^2 \prod_{j=1}^3 X_{ij}^{2/3+\varepsilon-\frac{1}{54}\delta} \right) \left( \prod_{j=1}^3 X_{3j}^{1-\frac{1}{54}\delta} \right). \end{aligned}$$

At this point we see why it is convenient to restrict to the set  $\mathcal{R}_\delta$ : the asymptotic formula of Proposition 1.2 provides a power saving with respect to the *largest* variable because of the inequality

$$\min_\ell X_\ell \geq \prod_\ell X_\ell^{\delta/9}.$$

Inserting the right-hand side of (5.10) into (5.9) yields a corresponding decomposition

$$(5.12) \quad N_{\Delta, T, \delta}(B) = N_{\Delta, T, \delta}^{(1)}(B) + E_{\Delta, T, \delta}(B).$$

In this section we estimate the error term. The bound (5.11) implies the bound

$$\int_{\mathcal{R}_\delta} \Psi_{\alpha_1, \alpha_2}(\mathbf{X}) \mathbf{X}^{\mathbf{v}-1} d\mathbf{X} \ll \delta^{-9} \prod_{i=1}^2 \prod_{j=1}^3 \alpha_{ij}^{1+\varepsilon}$$

that is valid subject to

$$\operatorname{Re}(v_{ij}) \geq \frac{2}{3} - \frac{\delta}{60} \quad (1 \leq i \leq 2, 1 \leq j \leq 3), \quad \operatorname{Re} v_{3j} \geq 1 - \frac{\delta}{60} \quad (1 \leq j \leq 3).$$

Let  $\sigma = \frac{1}{9} - \frac{\delta}{540}$ . Shifting all contours to  $\operatorname{Re} s_\ell = \sigma$ , we obtain

$$E_{\Delta, T, \delta}(B) \ll \frac{B^{1-\frac{\delta}{60}}}{\delta^9} \sum_{|\mathbf{b}|, |\mathbf{c}|, |\mathbf{f}|, |\mathbf{g}|, h \leq T} \left( \prod_{i=1}^2 \prod_{j=1}^3 \alpha_{ij}^{\frac{1}{3}+\varepsilon+\frac{\delta}{90}} \right) \int_{(\sigma)}^{(9)} \prod_\ell \frac{|v_\ell \widehat{f}_\Delta(s_\ell)|}{|1-2^{-v_\ell}|} |ds|.$$

Also with later applications in mind, we observe that for  $\operatorname{Re} s_\ell \geq 1/100$  the bounds (2.2) and (2.5) imply that

$$(5.13) \quad \mathcal{D} \left( \frac{v_\ell \widehat{f}_\Delta(s_\ell)}{1-2^{-v_\ell}} \right) \ll_{\mathcal{D}} \frac{\Delta^{18}}{|s_{11} s_{12} \cdots s_{33}|^2},$$

holds for any differential operator  $\mathcal{D}$  in the variables  $s_{11}, \dots, s_{33}$ . For now we use this with  $\mathcal{D} = \operatorname{id}$ , getting

$$E_{\Delta, T, \delta}(B) \ll B^{1-\frac{\delta}{60}} \delta^{-9} \Delta^{-18} T^{13+6(\frac{1}{3}+\varepsilon+\frac{\delta}{90})}.$$

In particular, we then have

$$(5.14) \quad E_{\Delta, T, \delta}(B) \ll B^{1-\frac{\delta}{60}} \delta^{-9} \Delta^{-18} T^{16},$$

uniformly for  $B, T, \delta, \Delta$  as in Lemma 5.1.

5.3. **The main term.** We insert the main term in (5.10) into (5.9) getting

$$N_{\Delta,T,\delta}^{(1)}(B) = \frac{1}{32} \sum_{|\mathbf{b}|, |\mathbf{c}|, |\mathbf{f}|, |\mathbf{g}|, h \leq T} \mu((\mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, h)) \int_{(1)}^{(9)} \frac{1}{\alpha^{\mathbf{v}}} \prod_{\ell} \frac{v_{\ell} \widehat{f}_{\Delta}(s_{\ell}) B^{s_{\ell}}}{1 - 2^{-v_{\ell}}} \\ \times \int_{\mathcal{R}_{\delta}} \mathcal{E}_{\alpha_1 \cdot \alpha_2} \cdot \mathcal{I}_{\alpha_1 \cdot \alpha_2}(\mathbf{X}_1, \mathbf{X}_2) X_{31} X_{32} X_{33} \mathbf{X}^{-\mathbf{v}-1} d\mathbf{X} \frac{d\mathbf{s}}{(2\pi i)^9}.$$

As a first step we would like to make this independent of  $\delta$  by replacing  $\mathcal{R}_{\delta}$  (defined in (5.8)) with the full range  $[1, \infty)^9$ . We write

$$\mathcal{R}_{\delta} = [1, \infty)^9 \setminus \mathcal{S}_{\delta}$$

and obtain a corresponding decomposition

$$(5.15) \quad N_{\Delta,T,\delta}^{(1)}(B) = N_{\Delta,T}^{(2)}(B) - N_{\Delta,T,\delta}^{(2)}(B).$$

We anticipate that  $N_{\Delta,T,\delta}^{(2)}(B)$  is small and quantify this in the following lemma.

**Lemma 5.2.** *We have*

$$N_{\Delta,T,\delta}^{(2)}(B) \ll T^{14} \Delta^{-9} \delta B (\log B)^4.$$

*Proof.* We first consider the  $\mathbf{s}$ -integral

$$\int_{(1)}^{(9)} \frac{1}{\alpha^{\mathbf{v}}} \prod_{\ell} \frac{v_{\ell} \widehat{f}_{\Delta}(s_{\ell}) B^{s_{\ell}}}{1 - 2^{-v_{\ell}}} \mathbf{X}^{-\mathbf{v}} \frac{d\mathbf{s}}{(2\pi i)^9} = \sum_{\mathbf{n} \in \mathbb{N}_0^9} \int_{(1)}^{(9)} (2^{\mathbf{n}} \cdot \alpha \cdot \mathbf{X})^{-\mathbf{v}} \prod_{\ell} (v_{\ell} \widehat{f}_{\Delta}(s_{\ell}) B^{s_{\ell}}) \frac{d\mathbf{s}}{(2\pi i)^9},$$

where of course  $2^{\mathbf{n}}$  is the vector  $(2^{n_{\ell}})_{\ell \in \{1,2,3\}^2}$ . By (2.8) and (5.7), the 9-fold inverse Mellin transform of  $\mathbf{s} \mapsto \prod_{\ell} v_{\ell} \widehat{f}_{\Delta}(s_{\ell})$  is a linear combination of functions of the type

$$\mathbf{x} \mapsto \prod_{\ell} \varphi_{\ell}(x_{\ell}), \quad \varphi_{\ell} = D^{\nu_{\ell}} f_{\Delta}$$

for  $\nu \in \mathbb{N}_0^9$  with  $|\nu|_1 = 9$ . Hence by Mellin inversion, the above 9-fold integral is a linear combination of expressions of the type

$$\sum_{\mathbf{n} \in \mathbb{N}_0^9} \tilde{F}_{\Delta,B}(2^{\mathbf{n}} \cdot \alpha \cdot \mathbf{X}),$$

where  $\tilde{F}$  is defined as in (5.2) but with some of the functions  $f_{\Delta}$  replaced with  $D^{\nu} f_{\Delta}$ . Invoking also the bounds of Lemma 2.6a and (2.28) along with the trivial bound  $(r_1; r_2; r_3) \leq (r_1 r_2 r_3)^{1/3}$ , it suffices to bound

$$T^{\varepsilon} \sum_{\mathbf{n} \in \mathbb{N}_0^9} \sum_{|\mathbf{b}|, |\mathbf{c}|, |\mathbf{f}|, |\mathbf{g}|, h \leq T} \int_{\mathcal{S}_{\delta}} \frac{|\tilde{F}_{\Delta,B}(2^{\mathbf{n}} \cdot \alpha \cdot \mathbf{X})|}{(X_{11} X_{12} X_{13} X_{21} X_{22} X_{23})^{1/3}} d\mathbf{X} \\ \ll T^{13+\varepsilon} \Delta^{-9} \sum_{\mathbf{n} \in \mathbb{N}_0^9} \int_{\mathcal{S}_{\delta}} \frac{F_{0,\tilde{B}(\mathbf{n})}(\mathbf{X})}{(X_{11} X_{12} X_{13} X_{21} X_{22} X_{23})^{1/3}} d\mathbf{X}$$

with  $\tilde{B}(\mathbf{n}) = B(1 + \Delta)2^{-|\mathbf{n}|_1}$ . Here we just used the simple observation that each  $\tilde{F}$  is of size  $O(\Delta^{-9})$  by (2.4) and the above remarks, and  $f_{\Delta}$  has support  $[0, 1 + \Delta]$ . We can further relax the integral by integrating over the slightly larger set  $\{\mathbf{X} \in [1, \infty)^9 \mid \min(X_{\ell}) \leq (2B)^{\delta}\}$ , so that the desired bound follows from Lemma 4.5 with  $H = (2B)^{\delta}$  and  $B = \tilde{B}(\mathbf{n})$ .  $\square$

We now focus on the main term  $N_{\Delta,T}^{(1)}(B)$  and introduce some notation. Let  $z_1, z_2 \in \mathbb{C}$ , and let  $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$  as in (5.7). Now define

$$(5.16) \quad \mathbf{w}_1 = \mathbf{v}_1 + (z_1, z_2, 1 - z_1 - z_2), \quad \mathbf{w}_2 = \mathbf{v}_2 + (z_1, z_2, 1 - z_1 - z_2), \quad \mathbf{w}_3 = \mathbf{v}_3,$$

and put  $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \mathbb{C}^9$  so that  $\mathbf{w}$  is a linear function in  $\mathbf{s}$  and  $\mathbf{z} = (z_1, z_2)$ . We use (1.9), (1.10) and Lemma 2.8 to write

$$N_{\Delta, T}^{(2)}(B) = \int_{(\frac{1}{3})} \int_{(\frac{1}{3})} \int_{(1)}^{(9)} \mathcal{G}_T(\mathbf{s}, \mathbf{z}) \Xi_{\Delta}(\mathbf{s}, \mathbf{z}) B^s \frac{d\mathbf{s}}{(2\pi i)^9} \frac{dz_1 dz_2}{(2\pi i)^2},$$

where

$$s = \sum_{\ell} s_{\ell},$$

and where

$$(5.17) \quad \mathcal{G}_T(\mathbf{s}, \mathbf{z}) = \sum_{|\mathbf{b}|, |\mathbf{c}|, |\mathbf{f}|, |\mathbf{g}|, h \leq T} \frac{\mu((\mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, h))}{\alpha_1^{\mathbf{w}_1} \alpha_2^{\mathbf{w}_2} \alpha_3^{\mathbf{w}_3}} \sum_q \frac{\varphi(q)}{q^3} \prod_{k=1}^3 (q; \alpha_{1k} \alpha_{2k}),$$

$$(5.18) \quad \Xi_{\Delta}(\mathbf{s}, \mathbf{z}) = \frac{2}{\pi} K(z_1) K(z_2) K(1 - z_1 - z_2) \prod_{\ell} \frac{v_{\ell} \widehat{f}_{\Delta}(s_{\ell})}{1 - 2^{-v_{\ell}}},$$

with  $K$  as in (2.27). Here we have quite a bit of flexibility for the  $\mathbf{s}$ -contours, we only need to make sure that we stay

$$(5.19) \quad \text{to the right of poles of } \widehat{f}_{\Delta}(s_{\ell})(1 - 2^{-v_{\ell}})^{-1}(w_{\ell} - 1)^{-1}.$$

This is the case, for instance, if  $\operatorname{Re} s_{\ell} > 1/9$  holds for all  $\ell$ . We make the following affine-linear change of variables in the  $\mathbf{s}$ -integral:

$$(5.20) \quad \begin{aligned} y_1 &= v_{11} - (1 - z_1), & y_2 &= v_{21} - (1 - z_1), \\ y_3 &= v_{12} - (1 - z_2), & y_4 &= v_{22} - (1 - z_2), & y_5 &= -1 + s = -1 + \sum_{i,j=1}^3 s_{ij}, \end{aligned}$$

and  $y_6, \dots, y_9$  are chosen to make the transformation unimodular, e.g.

$$(5.21) \quad y_6 = s_{11}, \quad y_7 = s_{12}, \quad y_8 = \frac{1}{2} s_{21}, \quad y_9 = \frac{1}{2} s_{22}.$$

We write  $A(\mathbf{y}) = \mathbf{s}$  for the corresponding inverse transformation  $A$ , whose Jacobian is 1. This gives

$$(5.22) \quad N_{\Delta, T}^{(2)}(B) = \int^{(11)} \frac{\mathcal{H}_{T, \Delta}(A(\mathbf{y}), \mathbf{z}) B^{1+y_5} d(\mathbf{y}, \mathbf{z})}{\mathcal{L}(\mathbf{y}) (2\pi i)^{11}}$$

with the lines of integration defined by

$$\operatorname{Re} z_j = 1/3, \quad \operatorname{Re} y_1 = \dots = \operatorname{Re} y_4 = \eta, \quad \operatorname{Re} y_5 = 5\eta, \quad \operatorname{Re} y_6 = \dots = \operatorname{Re} y_9 = 1/15,$$

with

$$\mathcal{H}_{T, \Delta}(A(\mathbf{y}), \mathbf{z}) = \mathcal{G}_T(A(\mathbf{y}), \mathbf{z}) \Xi_{\Delta}(A(\mathbf{y}), \mathbf{z})$$

and

$$\begin{aligned} \mathcal{L}(\mathbf{y}) &= \prod_{\ell} (w_{\ell} - 1) \\ &= y_1 y_2 y_3 y_4 (2y_5 - y_3 - y_1) (2y_5 - y_4 - y_2) (y_5 - y_2 + y_1) (y_5 - y_4 + y_3) (y_5 + y_4 - y_3 + y_2 - y_1), \end{aligned}$$

and  $\eta > 0$  is chosen so small (say  $\eta = 10^{-6}$ ) that we stay to the right of the poles of  $(w_{\ell} - 1)^{-1}$ . The lines of integration for  $y_6, \dots, y_9$  are to some extent arbitrary, for instance every line to the right of  $1/18$  and to the left of  $1/12$  satisfies  $\operatorname{Re} s_{\ell} > 0$  (as one can check by expressing  $s_{\ell}$  in terms of  $y_j$ ) and hence is in agreement with the condition (5.19). The fact that the integrand in (5.22) has 9 polar lines with 5 variables  $y_1, \dots, y_5$  shows that we can obtain at most  $9 - 5 = 4$  log-powers in the final asymptotic formula. By successive contour shifts we show the following asymptotic evaluation.



**Lemma 5.3.** *Let  $B \geq 1$ ,  $T \geq 1$ ,  $0 < \Delta < 1$ ,  $0 < \delta < 1/10$  and define*

$$(5.23) \quad c_{T,\Delta} = \frac{1}{24} \int^{(6)} \mathcal{H}_{T,\Delta} \left( A(\mathbf{y})|_{y_1=\dots=y_5=0}, \mathbf{z} \right) \frac{d(y_6, y_7, y_8, y_9, z_1, z_2)}{(2\pi i)^6},$$

*with  $\operatorname{Re} z_1 = \operatorname{Re} z_2 = 1/3$ ,  $\operatorname{Re} y_6 = \dots = \operatorname{Re} y_9 = 1/15$  as lines of integration. Then*

$$(5.24) \quad N_{\Delta,T}^{(2)}(B) = \frac{1}{24} c_{T,\Delta} B (\log B)^4 + O(T^{14} \Delta^{-18} B (\log B)^3).$$

The proof is a straightforward, but tedious computation that we postpone to the next section. Combining Lemma 5.3 with Lemma 5.2, (5.15), (5.12) and (5.14), we obtain

$$(5.25) \quad N_{\Delta,T,\delta}(B) = \frac{1}{24} c_{T,\Delta} B (\log B)^4 + O \left( \left( \frac{T}{\Delta \delta} \right)^{18} B^{1-\frac{\delta}{60}} + \Delta^{-18} T^{14} B (\log B)^3 (1 + \delta \log B) \right).$$

**5.4. Computation of the leading constant.** In this section we compute the constant  $c_{T,\Delta}$  defined in (5.23). First we observe that  $y_1 = \dots = y_5 = 0$  in combination with (5.20) and (5.7) implies

$$(5.26) \quad \mathbf{v}_1 = \mathbf{v}_2 = (1 - z_1, 1 - z_2, z_1 + z_2), \quad \mathbf{v}_3 = (1, 1, 1),$$

hence  $\mathbf{w}_1 = \mathbf{w}_2 = \mathbf{w}_3 = (1, 1, 1)$  by (5.16). Inserting this into (5.17), we conclude from Lemma 2.7 with  $\alpha = 3/4$ , say, that

$$\mathcal{G}_T \left( A(\mathbf{y})|_{y_1=\dots=y_5=0}, \mathbf{z} \right) = \mathcal{G}_T = C + O(T^{-\frac{3}{4}}),$$

where  $C$  is as in (1.3). Combining (2.27), (5.18), (5.26), and writing  $s_{ij}$  in terms of  $y_j$  by (5.7), (5.20) and (5.21), we find after a short calculation that

$$\Xi_{\Delta} \left( A(\mathbf{y})|_{y_1=\dots=y_5=0}, \mathbf{z} \right) = \frac{2}{\pi} \mathcal{K}(z_1, z_2) \mathcal{F}_{\Delta}(y_6, \dots, y_9, z_1, z_2) \left( \frac{1}{1 - \frac{1}{2}} \right)^3,$$

where

$$\begin{aligned} \mathcal{F}_{\Delta}(\mathbf{y}, \mathbf{z}) &= \widehat{f}_{\Delta}(y_6) \widehat{f}_{\Delta}(y_7) \widehat{f}_{\Delta} \left( \frac{1 - z_1}{2} - y_6 - y_7 \right) \widehat{f}_{\Delta}(2y_8) \widehat{f}_{\Delta}(2y_9) \widehat{f}_{\Delta} \left( \frac{1 - z_2}{2} - 2y_8 - 2y_9 \right) \\ &\quad \times \widehat{f}_{\Delta}(1 - z_1 - z_2 - y_6 - 2y_8) \widehat{f}_{\Delta}(z_2 - y_7 - 2y_9) \widehat{f}_{\Delta} \left( \frac{3z_1 + z_2 - 2}{2} + y_6 + y_7 + 2y_8 + 2y_9 \right) \end{aligned}$$

and

$$\mathcal{K}(z_1, z_2) = \Gamma(z_1) \cos \left( \frac{\pi z_1}{2} \right) \Gamma(z_2) \cos \left( \frac{\pi z_2}{2} \right) \Gamma(1 - z_1 - z_2) \cos \left( \frac{\pi(1 - z_1 - z_2)}{2} \right).$$

We would like to replace  $\mathcal{F}_{\Delta}(\mathbf{y}, \mathbf{z})$  with  $\mathcal{F}_0(\mathbf{y}, \mathbf{z})$  and estimate the corresponding error. For  $\operatorname{Re} z_j = 1/3$ ,  $\operatorname{Re} y_j = 1/15$ , Stirling's formula yields the crude bound  $\mathcal{K}(z_1, z_2) \ll |z_1 z_2|^{-1/6}$ , and (2.6) – (2.7) deliver the bound

$$\mathcal{F}_{\Delta}(\mathbf{y}, \mathbf{z}) - \mathcal{F}_0(\mathbf{y}, \mathbf{z}) \ll \Delta^{\frac{1}{20}} \left( \left| \frac{1 - z_2}{2} - 2y_8 - 2y_9 \right| \left| \frac{1 - z_1}{2} - y_6 - y_7 \right| |y_6 y_7 y_8 y_9| \right)^{-\frac{19}{20}}.$$

We now observe that for  $\operatorname{Re} z_j = 1/3$ ,  $\operatorname{Re} y_j = 1/15$  the integral

$$\begin{aligned} &\int^{(4)} \int^{(2)} |z_1 z_2|^{-1/6} \left( \left| \frac{1 - z_2}{2} - 2y_8 - 2y_9 \right| \left| \frac{1 - z_1}{2} - y_6 - y_7 \right| |y_6 y_7 y_8 y_9| \right)^{-\frac{19}{20}} |\mathbf{dz}| |\mathbf{dy}| \\ &\ll \int^{(4)} (|y_8 + y_9| |y_6 + y_7|)^{\frac{1}{20} - \frac{1}{6}} |y_6 y_7 y_8 y_9|^{-\frac{19}{20}} |\mathbf{dy}| = \left( \int_{(\frac{1}{15})} \int_{(\frac{1}{15})} |y_6 + y_7|^{-\frac{7}{60}} |y_6 y_7|^{-\frac{19}{20}} dy_6 dy_7 \right)^2 \end{aligned}$$

is absolutely convergent, and we conclude that

$$\begin{aligned} & \int^{(6)} \Xi_{\Delta} \left( A(\mathbf{y})|_{y_1=\dots=y_5=0}, \mathbf{z} \right) \frac{d(y_6, y_7, y_8, y_9, z_1, z_2)}{(2\pi i)^6} \\ &= \frac{16}{\pi} \int^{(6)} \mathcal{K}(z_1, z_2) \mathcal{F}_0(y_6, \dots, y_9, z_1, z_2) \frac{d(y_6, y_7, y_8, y_9, z_1, z_2)}{(2\pi i)^6} + O(\Delta^{1/20}) \end{aligned}$$

where we integrate over  $\operatorname{Re} z_j = 1/3$ ,  $\operatorname{Re} y_j = 1/15$ . We evaluate the  $\mathbf{y}$ -integral by Lemma 2.9 and the  $\mathbf{z}$ -integral by Lemma 2.10. The above discussion now delivers

$$\begin{aligned} (5.27) \quad c_{T,\Delta} &= \frac{1}{24} \cdot \frac{16}{\pi} \left( C + O(T^{-3/4}) \right) \left( \frac{\pi}{4} (\pi^2 - 3 + 24 \log 2) + O(\Delta^{1/20}) \right) \\ &= \frac{\pi^2 - 3 + 24 \log 2}{6} C + O \left( \Delta^{1/20} + T^{-3/4} \right). \end{aligned}$$

**5.5. The endgame.** We are ready to complete the proof of Theorem 1.1. Collecting (5.4), Lemma 5.1, (5.25) and (5.27), we find that

$$\begin{aligned} N_{\Delta}(B) &= N_{\Delta,T,\delta}(B) + O \left( B(\log B)^4 \left( \frac{1}{T^{1/4}} + T^{13}\delta \right) \right) \\ &= \frac{\pi^2 - 3 + 24 \log 2}{6 \cdot 24} C B(\log B)^4 \\ &\quad + O \left( B(\log B)^4 \left( \frac{1}{T^{1/4}} + \Delta^{1/20} + T^{14} \Delta^{-18} \left( \delta + \frac{1}{\log B} \right) \right) + \left( \frac{T}{\Delta \delta} \right)^{18} B^{1-\frac{\delta}{60}} \right). \end{aligned}$$

By (5.3), the passage from  $N_{\Delta}(B)$  to  $N(B)$  introduces an error  $\Delta B(\log B)^4$  that is already present in the above asymptotic formula, hence the same formula holds for  $N(B)$ . We now choose

$$(5.28) \quad \delta = \frac{1}{(\log B)^{99/100}}, \quad \Delta = \frac{1}{(\log B)^{1/24}}, \quad T = (\log B)^{1/120}.$$

to complete the proof of Theorem 1.1, but it remains to provide proofs for Lemmas 5.1 and 5.3.

**5.6. Proof of Lemma 5.1.** In order to estimate the difference between  $N_{\Delta,T,\delta}(B)$  and  $N_{\Delta,T}(B)$ , we reverse the steps between (5.5) and (5.9) and eventually use Lemma 4.4. Starting from (5.9), we have

$$\begin{aligned} N_{\Delta,T,\delta}(B) &= \frac{1}{32} \sum_{|\mathbf{b}|, |\mathbf{c}|, |\mathbf{f}|, |\mathbf{g}|, h \leq T} \mu((\mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, h)) \int_{(1)}^{(9)} \frac{1}{\alpha^{\mathbf{v}}} \prod_{\ell} \left( \frac{v_{\ell}}{1 - 2^{-v_{\ell}}} \widehat{f}_{\Delta}(s_{\ell}) B^{s_{\ell}} \right) \\ &\quad \times \left( \int_{\mathcal{R}_{\delta}} \sum_{\substack{\frac{1}{2} X_{\ell} < |x_{\ell}| \leq X_{\ell} \\ \ell \in \{1,2,3\}^2}} \chi(\alpha \cdot \mathbf{x}) \mathbf{X}^{-\mathbf{v}-1} d\mathbf{X} \right) \frac{ds}{(2\pi i)^9} \end{aligned}$$

with  $\alpha$  as in (2.15). For a vector  $\sigma \in \{0,1\}^9$  we define  $\mathbf{X}_{\sigma} = (2^{\sigma_{\ell}} X_{\ell})_{\ell \in \{1,2,3\}^2}$  and

$$\mathcal{R}_{\delta}(\sigma) = \{\mathbf{X} \in [1, \infty)^9 \mid \min(2^{\sigma_{\ell}} X_{\ell}) \geq \max(2^{\sigma_{\ell}} X_{\ell})^{\delta}\}.$$

By a change of variables, the integral over  $\mathcal{R}_{\delta}$  equals

$$\sum_{\sigma \in \{0,1\}^9} (-2)^{|\sigma|_1} \int_{\mathcal{R}_{\delta}(\sigma)} \sum_{0 < |x_{\ell}| \leq X_{\ell}} \chi(\alpha \cdot \mathbf{x}) \mathbf{X}_{\sigma}^{-\mathbf{v}-1} d\mathbf{X},$$

and by partial summation this equals

$$\left( \prod_{\ell} \frac{1}{v_{\ell}} \right) \sum_{\sigma \in \{0,1\}^9} (-1)^{|\sigma|_1} 2^{-\sum_{\ell} \sigma_{\ell} v_{\ell}} \sum_{\mathbf{x} \in \mathbb{Z}_0^9}^{(\sigma, \delta)} \frac{\chi(\alpha \cdot \mathbf{x})}{\mathbf{x}^{\mathbf{v}}},$$

where  $\sum_+^{(\sigma, \delta)}$  denotes a summation over  $\mathbf{x}$  satisfying

$$\min_{\ell}(2^{\sigma_{\ell}}|x_{\ell}|) \geq \max_{\ell}(2^{\sigma_{\ell}}|x_{\ell}|)^{\delta},$$

and correspondingly we write  $\sum_-^{(\sigma, \delta)}$  for a summation with the opposite condition

$$(5.29) \quad \min_{\ell}(2^{\sigma_{\ell}}|x_{\ell}|) < \max_{\ell}(2^{\sigma_{\ell}}|x_{\ell}|)^{\delta}.$$

In this notation, one has  $N_{\Delta, T}(B) - N_{\Delta, T, \delta}(B) \ll M_{\Delta, T, \delta}(B)$  where

$$M_{\Delta, T, \delta}(B) = \sum_{|\mathbf{b}|, |\mathbf{c}|, |\mathbf{f}|, |\mathbf{g}|, h \leq T} \sum_{\sigma \in \{0, 1\}^9} \left| \int_{(1)}^{(9)} 2^{-\sum_{\ell} \sigma_{\ell} v_{\ell}} \sum_{\mathbf{x} \in \mathbb{Z}_0^9}^{(\sigma, \delta)} \frac{\chi(\alpha \cdot \mathbf{x})}{\alpha^{\mathbf{v}} \mathbf{x}^{\mathbf{v}}} \prod_{\ell} \frac{\widehat{f}_{\Delta}(s_{\ell}) B^{s_{\ell}}}{1 - 2^{-v_{\ell}}} \frac{ds}{(2\pi i)^9} \right|.$$

We write the factor  $(1 - 2^{-v_{\ell}})^{-1}$  as a geometric series and apply Mellin inversion to recast the integral as

$$\sum_{\mathbf{k} \in \mathbb{N}_0^9} \sum_{\mathbf{x} \in \mathbb{Z}_0^9}^{(\sigma, \delta)} \chi(\alpha \cdot \mathbf{x}) F_{\Delta, B}(\alpha \cdot (2^{k_{\ell} + \sigma_{\ell}} x_{\ell})_{\ell}),$$

so that

$$M_{\Delta, T, \delta}(B) = \sum_{|\mathbf{b}|, |\mathbf{c}|, |\mathbf{f}|, |\mathbf{g}|, h \leq T} \sum_{\mathbf{x} \in \mathbb{Z}_0^9} \chi(\alpha \cdot \mathbf{x}) G_{\alpha}(\mathbf{x})$$

with

$$G_{\alpha}(\mathbf{x}) = \sum_{\sigma \in \{0, 1\}^9} \sum_{\mathbf{k} \in \mathbb{N}_0^9} F_{\Delta, B}(\alpha \cdot (2^{k_{\ell} + \sigma_{\ell}} x_{\ell})_{\ell}) \Psi_{\sigma, \delta}(\mathbf{x}),$$

in which  $\Psi_{\sigma, \delta}$  is the characteristic function of the set defined by (5.29). In particular,

$$\text{supp } G_{\alpha} \subseteq \{\mathbf{x} \in \mathbb{R}^9 \mid \min(|x_{\ell}|) \leq (2B(1 + \Delta))^{\delta}\}, \quad G_{\alpha}(\mathbf{x}) \ll \sum_{\mathbf{k} \in \mathbb{N}_0^9} F_{0, B(1 + \Delta)}(\alpha \cdot (2^{k_{\ell}} x_{\ell})_{\ell}).$$

By Lemma 4.4 with  $(2B(1 + \Delta))^{\delta}$  in place of  $H$  and  $B(1 + \Delta)$  in place of  $B$  we conclude that

$$M_{\Delta, T, \delta}(B) \ll \sum_{|\mathbf{b}|, |\mathbf{c}|, |\mathbf{f}|, |\mathbf{g}|, h \leq T} \sum_{\mathbf{k} \in \mathbb{N}_0^9} 2^{-(\frac{2}{3} - \varepsilon) \sum_{\ell} k_{\ell}} B(\log B)^3 \log(B^{\delta}) \ll \delta T^{13} B(\log B)^4,$$

as desired.

## 6. PROOF OF LEMMA 5.3

We start with the evaluation of  $N_{\Delta, T}^{(2)}(B)$ , defined in (5.22), and prove (5.24). We perform various contour shifts with the variables  $y_1, \dots, y_5$ . The variables  $y_6, \dots, y_9, z_1, z_2$  will be kept fixed. We will always stay in the region  $|\text{Re } y_1|, \dots, |\text{Re } y_5| \leq 10\eta$  with  $\eta = 10^{-6}$  as before, and we remember our choice  $\text{Re } z_1 = \text{Re } z_2 = 1/3$ . In this region we have  $\text{Re } w_{\ell} \geq 1 - 40\eta$ , as one can check from (5.7), (5.16), (5.20) and (5.21), and we derive now rather crude, but convenient bounds for the function  $\mathcal{H}_{T, \Delta}(\mathbf{s}, \mathbf{z})$  and its derivatives appearing in the integrand of (5.22) (the derivatives are needed for residue computations). Let  $\mathcal{D}_j$  denote a differential operator of degree  $j$  in  $s_{11}, \dots, s_{33}$ . Then for  $\text{Re } w_{\ell} \geq 1 - 40\eta$  we obtain by the most trivial estimates

$$(6.1) \quad \mathcal{D}_j \mathcal{G}_T(\mathbf{s}, \mathbf{z}) \ll_j \sum_{|\mathbf{b}|, |\mathbf{c}|, |\mathbf{f}|, |\mathbf{g}|, h \leq T} \prod_{k=1}^3 \alpha_{1k}^{1 - \text{Re } w_{1k} + \varepsilon} \alpha_{2k}^{1 - \text{Re } w_{2k} + \varepsilon} \ll T^{13 + 720\eta + \varepsilon} \ll T^{14}.$$

Similarly, for  $\text{Re } s_{\ell} > 1/100$  we conclude from (5.13) that

$$(6.2) \quad \mathcal{D}_j \Xi_{\Delta}(\mathbf{s}, \mathbf{z}) \ll_j \Delta^{-18} |s_{11} s_{12} \cdots s_{33} z_1 z_2|^{-2}.$$

We now shift successively the  $y_1, \dots, y_4$  contours to  $\text{Re } y_j = -j\eta$ ,  $1 \leq j \leq 4$ , thereby picking up a simple pole at 0 and a remaining integral. This leaves us altogether with 16 terms, some of which are identical by symmetry. We denote by  $V \subseteq \{y_1, y_2, y_3, y_4\}$  the set of variables that

have not been integrated out and distinguish several cases. For notational simplicity we write  $\tilde{\mathcal{H}}(y_1, \dots, y_5) = \mathcal{H}_{T, \Delta}(A(\mathbf{y}), \mathbf{z})$ .

**6.1. Case I:**  $V = \emptyset$ . The term consisting only of residues equals

$$\int_{(5\eta)} \frac{\tilde{\mathcal{H}}(0, 0, 0, 0, y_5) B^{y_5}}{4y_5^5} \frac{dy_5}{2\pi i} = \frac{1}{4} \tilde{\mathcal{H}}(\mathbf{0}) \frac{(\log B)^4}{4!} + O((\log B)^3 T^{14} \Delta^{-18})$$

by shifting the contour to  $\operatorname{Re} y_5 = -\eta$ , say, and spelling out the leading term of the residue, while estimating the lower order terms and the remaining integral trivially (and crudely) by (6.1) and (6.2).

**6.2. Case II:**  $V = \{y_1\}$ . Here we have

$$\int_{(-\eta)} \int_{(5\eta)} \frac{\tilde{\mathcal{H}}(y_1, 0, 0, 0, y_5) B^{y_5}}{2y_1 y_5^2 (y_5 - y_1)(2y_5 - y_1)(y_5 + y_1)} \frac{dy_5 dy_1}{(2\pi i)^2}.$$

Shifting the line  $\operatorname{Re} y_5 = 5\eta$  to  $\operatorname{Re} y_5 = -\eta/3$ , we pick up a pole at  $y_5 = -y_1$  and  $y_5 = 0$ , the latter of which as well as the remaining integral we estimate trivially. Hence the previous expression equals

$$\int_{(-\eta)} \frac{\tilde{\mathcal{H}}(y_1, 0, 0, 0, 0) B^{-y_1}}{12y_1^5} \frac{dy_1}{2\pi i} + O(T^{14} \Delta^{-18} \log B) = -\frac{1}{12} \tilde{\mathcal{H}}(\mathbf{0}) \frac{(\log B)^4}{4!} + O(T^{14} \Delta^{-18} (\log B)^3)$$

which we realize after shifting the line of integration to  $\operatorname{Re} y_1 = \eta$  and spelling out only the leading term of the residue at  $y_1 = 0$ . The same evaluation holds for  $V = \{y_2\}$ ,  $V = \{y_3\}$ ,  $V = \{y_4\}$ .

**6.3. Case IIIa:**  $V = \{y_1, y_2\}$ . In

$$\int_{(-\eta)} \int_{(-2\eta)} \int_{(5\eta)} \frac{\tilde{\mathcal{H}}(y_1, y_2, 0, 0, y_5) B^{y_5}}{y_1 y_2 y_5 (y_5 + y_1 - y_2)(y_5 - y_1 + y_2)(2y_5 - y_1)(2y_5 - y_2)} \frac{dy_5 dy_2 dy_1}{(2\pi i)^3}$$

we shift the line  $\operatorname{Re} y_5 = 5\eta$  to  $\operatorname{Re} y_5 = -\eta/3$  and argue similarly. Up to an error of  $O(T^{14} \Delta^{-18})$  coming from the simple pole at  $y_5 = 0$ , we pick up the residue at  $y_5 = y_1 - y_2$ , which equals

$$\int_{(-2\eta)} \int_{(-\eta)} \frac{\tilde{\mathcal{H}}(y_1, y_2, 0, 0, y_1 - y_2) B^{y_1 - y_2}}{2y_1 y_2 (y_1 - y_2)^2 (2y_1 - 3y_2)(y_1 - 2y_2)} \frac{dy_1 dy_2}{(2\pi i)^2} = O(T^{14} \Delta^{-18} \log B),$$

as we see from shifting  $\operatorname{Re} y_1$  to  $-5/2\eta$  and estimating trivially the contribution of the double pole at  $y_2 = y_1$ . The same bound holds by symmetry for  $V = \{y_3, y_4\}$ .

**6.4. Case IIIb:**  $V = \{y_1, y_3\}$ . In

$$\int_{(-\eta)} \int_{(-3\eta)} \int_{(5\eta)} \frac{\tilde{\mathcal{H}}(y_1, 0, y_3, 0, y_5) B^{y_5}}{2y_1 y_3 y_5 (y_5 + y_1)(y_5 - y_1 - y_3)(2y_5 - y_1 - y_3)(y_5 + y_3)} \frac{dy_5 dy_3 dy_1}{(2\pi i)^3}$$

we shift the line  $\operatorname{Re} y_5 = 5\eta$  to  $\operatorname{Re} y_5 = -\eta$ . Up to an error of  $O(T^{14} \Delta^{-18})$  for the simple pole at  $y_5 = 0$  we get contributions from two poles at  $y_5 = -y_1$  and  $y_5 = -y_3$ . The former yields

$$\int_{(-3\eta)} \int_{(-\eta)} \frac{\tilde{\mathcal{H}}(y_1, 0, y_3, 0, -y_1) B^{-y_1}}{2y_1^2 y_3 (y_1 - y_3)(2y_1 + y_3)(3y_1 + y_3)} \frac{dy_1 dy_3}{(2\pi i)^2} = O(T^{14} \Delta^{-18} \log B),$$

as one finds after shifting the line  $\operatorname{Re} y_1 = -\eta$  to  $\operatorname{Re} y_1 = \eta/2$  and estimating trivially the double pole at  $y_1 = 0$ . The latter yields

$$\int_{(-\eta)} \int_{(-3\eta)} \frac{\tilde{\mathcal{H}}(y_1, 0, y_3, 0, -y_3) B^{-y_3}}{2y_1 y_3^2 (y_3 - y_1)(y_1 + 2y_3)(y_1 + 3y_3)} \frac{dy_3 dy_1}{(2\pi i)^2}.$$

We shift the line  $\operatorname{Re} y_3 = -3\eta$  to  $\operatorname{Re} y_3 = \eta/4$ . Up to an error of  $O(T^{14} \Delta^{-18} \log B)$ , we get a contribution of the pole at  $y_3 = y_1$ , and its residue is

$$-\int_{(-\eta)} \frac{\tilde{\mathcal{H}}(y_1, 0, y_1, 0, -y_1) B^{-y_1}}{24y_1^5} \frac{dy_1}{2\pi i} = \frac{1}{24} \tilde{\mathcal{H}}(\mathbf{0}) \frac{(\log B)^4}{4!} + O(T^{14} \Delta^{-18} (\log B)^3).$$

6.5. **Case IIIc:**  $V = \{y_2, y_3\}$ . In

$$\int_{(-2\eta)} \int_{(-3\eta)} \int_{(5\eta)} \frac{\tilde{\mathcal{H}}(0, y_2, y_3, 0, y_5) B^{y_5}}{y_2 y_3 (2y_5 - y_2)(y_5 - y_2)(2y_5 - y_3)(y_5 + y_2 - y_3)(y_5 + y_3)} \frac{dy_5 dy_3 dy_2}{(2\pi i)^3}$$

we shift the line  $\operatorname{Re} y_5 = 5\eta$  to  $\operatorname{Re} y_5 = -\eta/2$ . We pick up one pole at  $y_5 = -y_3$  that contributes

$$\int_{(-2\eta)} \int_{(-3\eta)} \frac{\tilde{\mathcal{H}}(0, y_2, y_3, 0, -y_3) B^{-y_3}}{3y_2 y_3^2 (2y_3 - y_2)(y_2 + y_3)(y_2 + 2y_3)} \frac{dy_3 dy_2}{(2\pi i)^2}.$$

We shift the line  $\operatorname{Re} y_3 = -3\eta$  to  $\operatorname{Re} y_3 = \eta/2$ . The pole at  $y_3 = 0$  contributes  $O(T^{14} \Delta^{-18} \log B)$ , and the residue at  $y_3 = y_2/2$  equals

$$-\frac{2}{9} \int_{(-2\eta)} \frac{\tilde{\mathcal{H}}(0, y_2, y_2/2, 0, -y_2/2) B^{-y_2/2}}{y_2^5} \frac{dy_2}{2\pi i} = \frac{1}{72} \tilde{H}(0) \frac{(\log B)^4}{4!} + O(T^{14} \Delta^{-18} (\log B)^3),$$

as is readily confirmed by shifting the line of integration to the far right. The same evaluation holds for  $V = \{y_1, y_4\}$ .

6.6. **Case IIIId:**  $V = \{y_2, y_4\}$ . We consider

$$\int_{(-2\eta)} \int_{(-4\eta)} \int_{(5\eta)} \frac{\tilde{\mathcal{H}}(0, y_2, 0, y_4, y_5) B^{y_5}}{2y_2 y_4 y_5 (y_5 - y_2)(y_5 - y_4)(2y_5 - y_2 - y_4)(y_5 + y_2 + y_4)} \frac{dy_5 dy_4 dy_2}{(2\pi i)^3}.$$

We begin by moving  $\operatorname{Re} y_5 = 5\eta$  to  $\operatorname{Re} y_5 = -\eta$ . Observing the pole at  $y_5 = -y_2 - y_4$ , we then see that this integral equals

$$\int_{(-4\eta)} \int_{(-2\eta)} \frac{\tilde{\mathcal{H}}(0, y_2, 0, y_4, -y_2 - y_4) B^{-y_2 - y_4}}{6y_2 y_4 (y_2 + y_4)^2 (2y_2 + y_4)(y_2 + 2y_4)} \frac{dy_2 dy_4}{(2\pi i)^2} + O(T^{14} \Delta^{-18}).$$

Next, by shifting  $\operatorname{Re} y_2 = -2\eta$  to  $\operatorname{Re} y_2 = 5\eta$ , we pick up three residues and a remaining integral of size  $O(T^{14} \Delta^{-18} B^{-\eta})$ . The pole at  $y_2 = -y_4$  contributes  $O(T^{14} \Delta^{-18} \log B)$ , the pole at  $y_2 = 0$  gives

$$-\int_{(-4\eta)} \frac{\tilde{\mathcal{H}}(0, 0, 0, y_4, -y_4) B^{-y_4}}{12y_4^5} \frac{dy_4}{2\pi i} = \frac{1}{12} \tilde{\mathcal{H}}(\mathbf{0}) \frac{(\log B)^4}{4!} + O(T^{14} \Delta^{-18} (\log B)^3),$$

and the pole at  $y_2 = -y_4/2$  contributes

$$\frac{4}{9} \int_{(-4\eta)} \frac{\tilde{\mathcal{H}}(0, -y_4/2, 0, y_4 - y_4/2) B^{-y_4/2}}{y_4^5} \frac{dy_4}{2\pi i} = -\frac{1}{36} \tilde{\mathcal{H}}(\mathbf{0}) \frac{(\log B)^4}{4!} + O(T^{14} \Delta^{-18} (\log B)^3).$$

6.7. **Case IVa:**  $V = \{y_2, y_3, y_4\}$ . We wish to evaluate

$$\int^{(4)} \frac{\tilde{\mathcal{H}}(0, y_2, y_3, y_4, y_5) B^{y_5}}{y_2 y_3 y_4 (y_5 + y_3 - y_4)(y_5 + y_4 - y_3 + y_2)(y_5 - y_2)(y_3 - 2y_5)(y_2 + y_4 - 2y_5)} \frac{d(y_2, y_3, y_4, y_5)}{(2\pi i)^4},$$

with integrations over  $\operatorname{Re} y_2 = -2\eta$ ,  $\operatorname{Re} y_3 = -3\eta$ ,  $\operatorname{Re} y_4 = -4\eta$ ,  $\operatorname{Re} y_5 = 5\eta$ . First, we shift  $\operatorname{Re} y_5 = 5\eta$  to  $\operatorname{Re} y_5 = -\eta/2$ . The remaining integral is  $O(T^{14} \Delta^{-18} B^{-\eta/2})$  and we pick up a pole at  $y_5 = -y_2 + y_3 - y_4$  with residue

$$\int^{(3)} \frac{\tilde{\mathcal{H}}(0, y_2, y_3, y_4, -y_2 + y_3 - y_4) B^{-y_2 + y_3 - y_4}}{y_2 y_3 y_4 (y_2 - 2y_3 + 2y_4)(2y_2 - y_3 + y_4)(2y_2 - y_3 + 2y_4)(3y_2 - 2y_3 + 3y_4)} \frac{d(y_2, y_3, y_4)}{(2\pi i)^3},$$

here the integrations are over the same lines as before. Next, we shift  $\operatorname{Re} y_3 = -3\eta$  to  $\operatorname{Re} y_3 = -7\eta$ . The remaining integral then is  $O(T^{14} \Delta^{-18} B^{-\eta})$ , and we pick up a pole at  $y_3 = (2y_4 + y_2)/2$  with residue

$$-\frac{4}{3} \int_{(-4\eta)} \int_{(-2\eta)} \frac{\tilde{\mathcal{H}}(0, y_2, y_4 + y_2/2, y_4, -y_2/2) B^{-y_2/2}}{y_2^2 y_4 (2y_2 + y_4)(y_2 + 2y_4)(3y_2 + 2y_4)} \frac{dy_2 dy_4}{(2\pi i)^2} = O(T^{14} \Delta^{-18} \log B),$$

as is readily seen after shifting  $\operatorname{Re} y_2 = -2\eta$  to  $\operatorname{Re} y_2 = \eta$ . The same bound holds, by symmetry, for  $V = \{y_1, y_2, y_4\}$ .

6.8. **Case IVb:**  $V = \{y_1, y_3, y_4\}$ . In this case we consider

$$\int^{(4)} \frac{\tilde{\mathcal{H}}(y_1, 0, y_3, y_4, y_5) B^{y_5}}{y_1 y_3 y_4 (y_3 - y_4 + y_5)(y_5 - y_1 - y_3 + y_4)(y_1 + y_5)(y_4 - 2y_5)(y_1 + y_3 - 2y_5)} \frac{d(y_1, y_3, y_4, y_5)}{(2\pi i)^4},$$

with integrations over the lines  $\operatorname{Re} y_1 = -\eta$ ,  $\operatorname{Re} y_3 = -3\eta$ ,  $\operatorname{Re} y_4 = -4\eta$ ,  $\operatorname{Re} y_5 = 5\eta$ . As in the previous case, we shift  $\operatorname{Re} y_5 = 5\eta$  to  $\operatorname{Re} y_5 = -\eta/2$ . The remaining integral is  $O(T^{14}\Delta^{-18}B^{-\eta/2})$ , the pole at  $y_5 = y_1 + y_3 - y_4$  contributes  $O(T^{14}\Delta^{-18})$ , and we are left with the pole at  $y_5 = -y_1$ . The latter has residue

$$\int^{(3)} \frac{\tilde{\mathcal{H}}(y_1, 0, y_3, y_4, -y_1) B^{-y_1}}{y_1 y_3 y_4 (y_1 - y_3 + y_4)(2y_1 + y_3 - y_4)(2y_1 + y_4)(3y_1 + y_3)} \frac{d(y_1, y_3, y_4)}{(2\pi i)^3},$$

with lines of integrations as before. We shift  $\operatorname{Re} y_1 = -\eta$  to  $\operatorname{Re} y_1 = \eta/2$ . The remaining integral is  $O(T^{14}\Delta^{-18}B^{-\eta/2})$ , the pole at  $y_1 = 0$  contributes  $O(T^{14}\Delta^{-18})$ , and we only need to consider the pole at  $y_1 = (y_4 - y_3)/2$  whose residue contributes

$$-\frac{4}{3} \int_{(-4\eta)} \int_{(-3\eta)} \frac{\tilde{\mathcal{H}}((y_4 - y_3)/2, 0, y_3, y_4, (y_3 - y_4)/2) B^{(y_3 - y_4)/2}}{(y_3 - y_4)^2 (y_3 - 3y_4)(y_3 - 2y_4)} \frac{dy_3 dy_4}{(2\pi i)^2}.$$

Now we shift  $\operatorname{Re} y_3 = -3\eta$  to  $\operatorname{Re} y_3 = -5\eta$ . The remaining integral is  $O(T^{14}\Delta^{-18}B^{-\eta})$ , and the pole at  $y_3 = y_4$  contributes  $O(T^{14}\Delta^{-18} \log B)$ . The same bound holds in the case  $V = \{y_1, y_2, y_3\}$ .

6.9. **Case V:**  $V = \{y_1, y_2, y_3, y_4\}$ . Finally, in the case where none of the variables has been integrated out, we shift  $\operatorname{Re} y_5 = 5\eta$  to  $-\eta/2$ ; the remaining integral is  $O(T^{14}\Delta^{-18}B^{-\eta/2})$ , and we pick up a pole with residue

$$\int^{(4)} \frac{\tilde{\mathcal{H}}(y_1, y_2, y_3, y_4, y_1 - y_2 + y_3 - y_4) B^{y_1 - y_2 + y_3 - y_4}}{y_1 y_2 y_3 y_4 (y_1 - y_2 + 2y_3 - 2y_4)(2y_1 - 2y_2 + y_3 - y_4)} \times \frac{1}{(2y_1 - 3y_2 + 2y_3 - 3y_4)(y_1 - 2y_2 + y_3 - 2y_4)} \frac{d(y_1, y_2, y_3, y_4)}{(2\pi i)^4}.$$

Here, all lines of integration are given by  $\operatorname{Re} y_j = -j\eta$ . We shift  $\operatorname{Re} y_1$  to  $-7/2\eta$ . The remaining integral is  $O(T^{14}\Delta^{-18}B^{-\eta/2})$ , and we pick up a simple pole at  $y_1 = y_2 - (y_3 - y_4)/2$  with residue

$$-\frac{4}{3} \int^{(3)} \frac{\tilde{\mathcal{H}}(y_2 - (y_3 - y_4)/2, y_2, y_3, y_4, (y_3 - y_4)/2) B^{(y_3 - y_4)/2}}{y_2 y_3 y_4 (y_3 - y_4)(-2y_2 + y_3 - y_4)(-y_2 + y_3 - 2y_4)(-2y_2 + y_3 - 3y_4)} \frac{d(y_2, y_3, y_4)}{(2\pi i)^3},$$

with lines of integration still given by  $\operatorname{Re} y_j = -j\eta$ . Next we shift the line for  $y_3$  to  $\operatorname{Re} y_3 = 5\eta$ . The remaining integral is  $O(T^{14}\Delta^{-18}B^{-\eta})$ , and the pole at  $y_3 = y_4$  contributes  $O(T^{14}\Delta^{-18})$ .

Summarizing all previous calculations, we obtain (5.24) from Cases I, II (with multiplicity 4), IIIb, IIIc (with multiplicity 2) and IIId, since

$$\frac{1}{4} - \frac{4}{12} + \frac{1}{24} + \frac{2}{72} + \frac{1}{12} - \frac{1}{36} = \frac{1}{24}.$$

## 7. THE GEOMETRY OF THE CREPANT RESOLUTION

Let  $X \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$  be the smooth triprojective variety described in (1.6) with trihomogeneous coordinates  $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (x_1, x_2, x_3; y_1, y_2, y_3; z_1, z_2, z_3)$ . The aim of this chapter is to compute Peyre's alpha invariant of  $X$ . We will not specify the base field as the results in this chapter are purely algebraic and independent of the base field.

Along with  $X$  we consider the non-singular biprojective surface  $Y \subset \mathbb{P}^2 \times \mathbb{P}^2$  defined in bihomogeneous coordinates  $(\mathbf{y}, \mathbf{z})$  by  $y_1 z_1 = y_2 z_2 = y_3 z_3$ , and the subvariety  $Z \subset \mathbb{P}^2 \times \mathbb{P}^2$  defined in bihomogeneous coordinates  $(\mathbf{x}, \mathbf{z})$  by  $x_1 z_1 + x_2 z_2 + x_3 z_3 = 0$ . We also recall that  $V \subset \mathbb{P}^2 \times \mathbb{P}^2$  is the

singular biprojective cubic threefold with bihomogeneous coordinates  $(\mathbf{x}, \mathbf{y}) = (x_1, x_2, x_3; y_1, y_2, y_3)$  as in (1.1). There are natural projections

$$p : X \rightarrow Y, \quad g : X \rightarrow Z, \quad f : X \rightarrow V$$

defined by  $p : (\mathbf{x}, \mathbf{y}, \mathbf{z}) \mapsto (\mathbf{y}, \mathbf{z})$ ,  $g : (\mathbf{x}, \mathbf{y}, \mathbf{z}) \mapsto (\mathbf{x}, \mathbf{z})$ ,  $f : (\mathbf{x}, \mathbf{y}, \mathbf{z}) \mapsto (\mathbf{x}, \mathbf{y})$ . We will frequently use these maps and its corresponding induced functorial maps. We will also use the  $\mathbb{G}_m^3$ -action on  $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$  defined by

$$\gamma(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\gamma_1 x_1, \gamma_2 x_2, \gamma_3 x_3; \gamma_1 y_1, \gamma_2 y_2, \gamma_3 y_3; \gamma_1^{-1} z_1, \gamma_2^{-1} z_2, \gamma_3^{-1} z_3)$$

for  $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{G}_m^3$ , and its restriction to  $\mathbb{G}_m^3$ -actions on  $X$ ,  $\mathbb{P}^2 \times \mathbb{P}^2$  and  $Z$ , the latter two given by

$$(7.1) \quad \gamma(\mathbf{x}, \mathbf{z}) = (\gamma_1 x_1, \gamma_2 x_2, \gamma_3 x_3; \gamma_1^{-1} z_1, \gamma_2^{-1} z_2, \gamma_3^{-1} z_3).$$

The morphism  $g$  is then  $\mathbb{G}_m^3$ -equivariant and the base extension of the morphism  $h : Y \rightarrow \mathbb{P}^2$ ,  $(\mathbf{y}, \mathbf{z}) \mapsto \mathbf{z}$  along the second projection  $\text{pr}_2 : Z \rightarrow \mathbb{P}^2$ . As  $h$  is the blow-up of  $\mathbb{P}^2$  at the three points where two of the  $\mathbf{z}$ -coordinates vanish, we thus obtain that  $g$  is the blow-up at the union of the three disjoint lines  $l_i$  on  $Z$  defined by

$$(7.2) \quad l_i : x_i = z_j = z_k = 0$$

for  $\{i, j, k\} = \{1, 2, 3\}$ .

**7.1. The pseudoeffective cone.** Nine integral subsurfaces of  $X$  will be important for the computation of  $\alpha(X)$ : if  $1 \leq i \leq 3$  and  $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$ , then

$D_i \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$  is defined by the equations  $x_i = y_i = z_j = z_k = 0$ ,

$E_i \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$  is defined by the equations  $x_j z_j + x_k z_k$  and  $y_j = y_k = z_i = 0$ ,

$F_i \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$  is defined by the equations  $x_i = x_j y_k + x_k y_j = x_j z_j + x_k z_k = 0$  and  $y_j z_j - y_i z_i = y_k z_k - y_i z_i = 0$ .

Here  $D_i$  is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$  while  $E_i$  is a  $\mathbb{P}^1$ -bundle over the line in  $\mathbb{P}^2 \times \mathbb{P}^2$  with coordinates  $(\mathbf{y}, \mathbf{z})$  defined by  $y_j = y_k = z_i = 0$ . For  $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in F_i$ , we note that one of the two equalities  $(x_j, x_k) = (y_j, -y_k)$  or  $(x_j, x_k) = (z_k, -z_j)$  holds in  $\mathbb{P}^1$ . Hence  $p : X \rightarrow Y$  restricts to an isomorphism from  $F_i$  to  $Y$ .

The  $\alpha$ -invariant is defined by means of Cartier divisors. As  $X$  is smooth, we may also view such divisors as Weil divisors [13, p. 141] and regard them as members of the free abelian group  $\text{Div } X$  generated by the prime divisors. We may then extend this group to the group  $\text{Div}_{\mathbb{R}} X = \text{Div } X \otimes_{\mathbb{Z}} \mathbb{R}$  of  $\mathbb{R}$ -divisors and consider the submonoid of effective  $\mathbb{R}$ -divisors (see [15, p. 48]). The pseudoeffective cone  $C_{\text{eff}}(X) \subset \text{Pic } X \otimes \mathbb{R}$  is the closure of the convex cone spanned by the classes of all effective  $\mathbb{R}$ -divisors on  $X$  (see [15, p. 47]). The main result of this subsection is Proposition 7.1 below, asserting that  $C_{\text{eff}}(X)$  is spanned by the nine classes  $[D_i], [E_i], [F_i]$ ,  $1 \leq i \leq 3$ . We start with the following lemma.

**Lemma 7.1.** *The group  $\mathbb{G}_m^3$  acts trivially on  $\text{Pic } X$ .*

*Proof.* The surface  $Y$  is a del Pezzo surface of degree six and  $\text{Pic } Y$  is spanned by the classes of its six lines. The image  $p^*(\text{Pic } Y)$  of the functorial map  $p^* : \text{Pic } Y \rightarrow \text{Pic } X$  is therefore spanned by the classes of all  $D_i$  and  $E_i$ . As  $D_i$  and  $E_i$  are  $\mathbb{G}_m^3$ -invariant,  $\mathbb{G}_m^3$  acts trivially on  $p^*(\text{Pic } Y)$ .

Next, let  $L = \text{pr}_1^*(\mathcal{O}_{\mathbb{P}^2}(1))$  be the sheaf associated to the first projection  $\text{pr}_1$  of  $X \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ . Then,  $[L] + p^*(\text{Pic } Y)$  generates  $\text{Pic } X / p^*(\text{Pic } Y) \cong \mathbb{Z}$  as  $X$  is a  $\mathbb{P}^1$ -bundle over  $Y$ . As  $\mathbb{G}_m^3$  acts trivially on  $[L]$  and  $p^*(\text{Pic } Y)$ , it acts trivially on  $\text{Pic } X$ .  $\square$

The following lemma will make it easier to determine  $C_{\text{eff}}(X)$ .

**Lemma 7.2.** *An effective divisor on  $X$  is linearly equivalent to a  $\mathbb{G}_m^3$ -invariant effective divisor on  $X$ .*



*Proof.* An effective divisor  $D$  on  $X$  is given by the vanishing of a global section of the invertible  $\mathcal{O}_X$ -module  $L = \mathcal{O}_X(D)$ . As  $\mathbb{G}_m^3$  stabilizes the class of  $L$  in  $\text{Pic } X$ , it therefore follows from the proof of [17, Prop. 1.5, p. 34] that we may endow  $L$  with a  $\mathbb{G}_m^3$ -linearization (see also [13, Prop. 2.3]). This is equivalent to a lifting of the  $\mathbb{G}_m^3$ -action on  $X$  to a  $\mathbb{G}_m^3$ -action on the line bundle  $\mathbf{L} \rightarrow X$  defined by  $L$  (see [17, p. 31]). There is thus an induced rational representation of  $\mathbb{G}_m^3$  on  $H^0(X, L)$ . Since  $\mathbb{G}_m^3$  is diagonalizable ([25, p. 21]), this induced rational representation must be a direct sum of one-dimensional ones. Hence there is a  $\mathbb{G}_m^3$ -invariant one-dimensional subspace  $S$  of  $H^0(X, L)$ . The divisor of zeros of  $S$  ([13, p. 157]) is then a  $\mathbb{G}_m^3$ -invariant effective divisor on  $X$  linearly equivalent to  $D$ , as desired.  $\square$

In the following we will use  $\mathbb{G}_m^3$ -linearizations on invertible sheaves  $L$  on  $\mathbb{P}^2 \times \mathbb{P}^2$  and  $Z$  compatible with (7.1). To construct such a linearization on  $L = \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(m, n)$ , let  $\langle m, n \rangle = \binom{m+3}{3} \binom{n+3}{3} - 1$  and

$$h_{m,n} : \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^{\langle m,n \rangle}$$

be the morphism defined by all monomials of bidegree  $(m, n)$  in  $(x_1, x_2, x_3; z_1, z_2, z_3)$ . Then we have  $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(m, n) = h_{m,n}^* \mathcal{O}_{\mathbb{P}^{\langle m,n \rangle}}(1)$ , and there is a natural  $\mathbb{G}_m^3$ -action on  $H^0(\mathbb{P}^{\langle m,n \rangle}, \mathcal{O}_{\mathbb{P}^{\langle m,n \rangle}}(1))$  given by

$$(7.3) \quad (\gamma G)(\mathbf{x}, \mathbf{z}) = G(\gamma_1 x_1, \gamma_2 x_2, \gamma_3 x_3; \gamma_1^{-1} z_1, \gamma_2^{-1} z_2, \gamma_3^{-1} z_3)$$

for homogeneous polynomials  $G(\mathbf{x}, \mathbf{z})$  of bidegree  $(m, n)$ . This  $\mathbb{G}_m^3$ -action gives rise to a  $\mathbb{G}_m^3$ -linearization on  $\mathcal{O}_{\mathbb{P}^2}(1)$ , which may be pulled back to a  $\mathbb{G}_m^3$ -linearization on  $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(m, n) = h_{m,n}^* \mathcal{O}_{\mathbb{P}^{\langle m,n \rangle}}(1)$  (see [17, Prop. 1.7, p. 34]). Similarly, by considering the restriction of  $h_{m,n}$  to  $Z$ , we obtain a  $\mathbb{G}_m^3$ -linearization on  $\mathcal{O}_Z(m, n)$  such that the induced restriction from  $H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(m, n))$  to  $H^0(Z, \mathcal{O}_Z(m, n))$  is  $\mathbb{G}_m^3$ -equivariant.

**Lemma 7.3.** *Let  $\Delta$  be a  $\mathbb{G}_m^3$ -invariant effective divisor on  $Z$ . Then there exists a one-dimensional  $\mathbb{G}_m^3$ -invariant subspace  $S$  of  $H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(m, n))$  such that  $\Delta$  is the divisor of the section  $s_Z \in H^0(Z, \mathcal{O}_Z(m, n))$  for any  $s \in S \setminus \{0\}$ .*

*Proof.* Every effective divisor on  $Z$  is the divisor  $\text{div}(\sigma)$  of some global section  $\sigma$  of an invertible sheaf  $L$  on  $Z$ . It is well known (cf. e.g. [23, Th. 2.4]) that any invertible sheaf on  $Z$  is isomorphic to some  $\mathcal{O}_Z(m, n)$  where  $m, n \geq 0$  whenever  $H^0(Z, \mathcal{O}_Z(m, n)) \neq 0$ . We may and shall thus assume that  $L = \mathcal{O}_Z(m, n)$  for  $m, n \geq 0$ . An effective divisor  $\Delta$  on  $Z$  will then correspond to a one-dimensional subspace  $\Sigma$  of  $H^0(Z, \mathcal{O}_Z(m, n))$  for some  $m, n \geq 0$  (cf. [13, p. 157]), and  $\Delta$  will be  $\mathbb{G}_m^3$ -invariant if and only if  $\Sigma$  is  $\mathbb{G}_m^3$ -invariant.

We now apply the Künneth formula in [22] to  $\text{pr}_1^*(\mathcal{O}_{\mathbb{P}^2}(k)) \otimes \text{pr}_2^*(\mathcal{O}_{\mathbb{P}^2}(l))$ . We then obtain that  $H^1(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(k, l)) = 0$  as  $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k)) = 0$  for all  $k$ . In particular, we conclude that  $H^1(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(m-1, n-1)) = 0$ , and hence the restriction map from  $H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(m, n))$  to  $H^0(Z, \mathcal{O}_Z(m, n))$  must be surjective.

As this restriction map is  $\mathbb{G}_m^3$ -equivariant and the  $\mathbb{G}_m^3$ -representation on  $H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(m, n))$  is a direct sum of one-dimensional ones, there is some one-dimensional  $\mathbb{G}_m^3$ -invariant subspace of  $H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(m, n))$ , which restricts to  $\Sigma$  on  $Z$ .  $\square$

We prepare for the statement of the next lemma with a definition. A bihomogeneous polynomial  $G(\mathbf{x}, \mathbf{z})$  of bidegree  $(m, n)$  is said to be an *eigenpolynomial* under  $\mathbb{G}_m^3$  if it is contained in a one-dimensional  $\mathbb{G}_m^3$ -invariant subspace of  $H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(m, n))$ . In other words,  $G$  is an eigenpolynomial if and only if for each  $\gamma \in \mathbb{G}_m^3$ , we can find a constant  $c$  such that  $\gamma G = cG$  under the action in (7.3). In the following lemma, monomials in three variables occur. We write these in the compact form  $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} x_3^{a_3}$  for  $\mathbf{a} \in \mathbb{N}_0^3$ . Confusion with the notation (1.13) that was used in the analytic part should not arise.

**Lemma 7.4.** *Let  $G(\mathbf{x}, \mathbf{z})$  be a bihomogeneous eigenpolynomial under  $\mathbb{G}_m^3$ . Then there exists a monomial  $M_0 = \mathbf{x}^{\mathbf{e}} \mathbf{z}^{\mathbf{f}}$  and a ternary homogeneous polynomial  $H$  such that  $G = M_0 H(x_1 z_1, x_2 z_2, x_3 z_3)$ .*



*Proof.* Let  $I$  be the set of all sextuples  $(a_1, a_2, a_3, b_1, b_2, b_3)$  such that  $\mathbf{x}^{\mathbf{a}}\mathbf{z}^{\mathbf{b}}$  is a monomial in  $G$  with non-zero coefficient. As  $\gamma M = \gamma^{\mathbf{a}-\mathbf{b}} M$  for  $M = \mathbf{x}^{\mathbf{a}}\mathbf{z}^{\mathbf{b}}$ , the characters sending  $\gamma \in \mathbb{G}_m^3$  to  $\gamma^{\mathbf{a}-\mathbf{b}} \in \mathbb{G}_m$  coincide for all sextuples in  $I$ . The triples  $(a_1 - b_1, a_2 - b_2, a_3 - b_3)$  and the sextuples  $(e_1, e_2, e_3, f_1, f_2, f_3)$  with

$$e_i = \max(a_i - b_i, 0), \quad f_i = \max(b_i - a_i, 0)$$

will thus be the same for all sextuples in  $I$ . Hence, defining  $M_0 = \mathbf{x}^{\mathbf{e}}\mathbf{z}^{\mathbf{f}}$ , we get that  $M = M_0 \prod_i (x_i z_i)^{\min(a_i, b_i)}$  for any monomial  $M = \mathbf{x}^{\mathbf{a}}\mathbf{z}^{\mathbf{b}}$  in  $G$  with non-zero coefficient. Thus there exists a homogeneous polynomial  $H$  of degree

$$\sum_{i=1}^3 \min(a_i, b_i) = \frac{1}{2} \sum_{i=1}^3 (a_i + b_i - e_i - f_i) \geq 0$$

with  $G = M_0 H(x_1 z_1, x_2 z_2, x_3 z_3)$ .  $\square$

We now consider the images  $g^*(\Delta) \in \text{Div } X$  of effective divisors  $\Delta \in \text{Div } Z$  under the functorial map  $g^* : \text{Div } Z \rightarrow \text{Div } X$ .

**Lemma 7.5.** *Let  $H(t_1, t_2, t_3)$  be a ternary homogeneous polynomial of degree  $n$  not divisible by  $t_1 + t_2 + t_3$ , and let  $\Delta \in \text{Div } Z$  be the effective divisor defined by  $H(x_1 z_1, x_2 z_2, x_3 z_3)$ . Then the multiplicity of  $D_i$  in  $g^*(\Delta) \in \text{Div } X$  is equal to  $n$  for  $i = 1, 2$  and  $3$ .*

*Proof.* Let  $Z_0 \subset Z$  be the subscheme associated to  $\Delta$  (cf. [13, p. 145]) and let  $l_1, l_2, l_3$  be the three lines on  $Z$  described in (7.2). Then,  $D_1 + D_2 + D_3$  is the exceptional divisor (cf. [11, App. B6]) of the blow-up  $g : X \rightarrow Z$  at  $l_1 \cup l_2 \cup l_3$ . Therefore, the multiplicity of  $D_i$  in  $g^*(\Delta)$  must be equal to the multiplicity  $m_i$  of  $Z_0$  along  $l_i$  (see [11, p. 79] for the definition of  $m_i$  and [13, Ch. 5, Prop. 5.3] for the proof of a similar statement).

It suffices to prove the assertion for  $D_3$  and we may also use the equation  $x_1 z_1 + x_2 z_2 + x_3 z_3 = 0$  for  $Z$  to eliminate  $t_3 = -t_1 - t_2$ . This replaces  $H(t_1, t_2, t_3)$  by a non-zero binary form  $G(t_1, t_2)$ . Then,  $Z_0$  is the subscheme of  $\mathbb{P}^2 \times \mathbb{P}^2$  defined by  $(x_1 z_1 + x_2 z_2 + x_3 z_3, G(x_1 z_1, x_2 z_2))$  and  $l_3$  the subscheme defined by  $(z_1, z_2, x_3)$ . It is now clear from the definition of  $m_3$  that  $m_3 = n$ , as  $G(x_1 z_1, x_2 z_2)$  is of degree  $n$  with respect to  $(z_1, z_2)$ .  $\square$

We are now in a position to determine  $C_{\text{eff}}(X)$ . Recall that  $\text{Pic } X$  is a free abelian group of rank five ([4, Theorem 4]).

**Proposition 7.1.** *The pseudoeffective cone  $C_{\text{eff}}(X)$  is spanned by the nine classes  $[D_i], [E_i], [F_i]$ ,  $1 \leq i \leq 3$ .*

*Proof.* By Lemma 7.2, it is enough to show that the class  $[D] \in \text{Pic } X$  of any  $\mathbb{G}_m^3$ -invariant effective divisor  $D$  on  $X$  is in the cone spanned by the nine classes above. To do this, it suffices to treat the case where none of  $D_1, D_2, D_3$  occur in the prime decomposition of  $D$  as  $D_1, D_2$  and  $D_3$  are  $\mathbb{G}_m^3$ -invariant.

Now let  $D_U$  be the restriction of  $D$  to  $U = X \setminus \cup_{i=1}^3 D_i$  and let  $g_U : U \rightarrow Z$  be the restriction of  $g$  to  $U$ . Then  $g_U$  is an open immersion with  $Z \setminus g_U(U)$  of codimension two in  $Z$ . The functorial map  $g_U^* : \text{Div } Z \rightarrow \text{Div } U$  is thus an isomorphism, which restricts to an isomorphism between the submonoids of  $\mathbb{G}_m^3$ -invariant effective divisors on  $Z$  and  $U$ . There are therefore a unique  $\mathbb{G}_m^3$ -invariant effective divisor  $\Delta$  on  $Z$  with  $g_U^*(\Delta) = D_U$  and unique non-negative integers  $n_i$  with  $g^*(\Delta) = D + \sum_{i=1}^3 n_i D_i$ .

By Lemma 7.3 and Lemma 7.4 there is a decomposition  $\Delta = \Delta' + \Delta''$  into two  $\mathbb{G}_m^3$ -invariant effective divisors on  $Z$  where  $\Delta'$  is defined by a monomial  $\mathbf{x}^{\mathbf{e}}\mathbf{y}^{\mathbf{f}}$  and  $\Delta''$  by  $H(x_1 z_1, x_2 z_2, x_3 z_3)$  for a ternary form  $H(t_1, t_2, t_3)$ . As the divisors of  $x_i$  (resp.  $z_i$ ) are given by  $D_i + F_i$  (resp.  $D_j + D_k + E_i$ ), we infer that

$$g^*(\Delta') = \sum_{i=1}^3 e_i (D_i + F_i) + \sum_{i=1}^3 f_i (D_j + D_k + E_i).$$

By Lemma 7.5 we have also a decomposition  $g^*(\Delta'') = n(D_1 + D_2 + D_3) + D^*$  where  $n = \deg H$  and  $D^*$  is an effective divisor where  $D_1, D_2$  and  $D_3$  do not occur. By adding these two decompositions and comparing the result with  $g^*(\Delta) = D + \sum_{i=1}^3 n_i D_i$ , we obtain that

$$D = D^* + \sum_{i=1}^3 e_i F_i + \sum_{i=1}^3 f_i E_i.$$

Moreover, as  $g^*(\Delta'')$  is linearly equivalent to the divisor  $n(D_i + F_i) + n(D_j + D_k + E_i)$  of  $x_i^n z_i^n$ , we get that  $[D^*] = n[E_i + F_i]$  for any  $i \in \{1, 2, 3\}$  and that  $[D]$  belongs to the cone spanned by  $[E_1], [E_2], [E_3], [F_1], [F_2]$  and  $[F_3]$ .  $\square$

**7.2. Computation of  $\alpha(X)$ .** In this section we compute Peyre's  $\alpha$ -invariant (see [18, Def. 2.4]) for  $X$ . To do this, we let  $D_0$  (resp.  $D_4$ ) be the effective divisors given by  $L(\mathbf{z}) = 0$  (resp.  $M(\mathbf{x}) = 0$ ) for two fixed ternary linear forms  $L$  and  $M$ . We then have the following linear equivalences

$$(7.4) \quad E_i \sim D_0 - D_j - D_k$$

as  $\text{div}(z_i = 0) \sim D_0$ , and

$$(7.5) \quad F_i \sim D_4 - D_i$$

as  $\text{div}(x_i = 0) \sim D_4$ .

**Lemma 7.6.** *The divisor  $2D_0 - D_1 - D_2 - D_3 + 2D_4$  is an anticanonical divisor on  $X$ .*

*Proof.* The canonical sheaf  $\omega_V$  is isomorphic to  $\mathcal{O}_V(-2, -1)$  as  $V$  is of bidegree  $(1, 2)$ . Further, by [4, Theorem 4] we have that  $\omega_X = f^*\omega$  for the morphism  $f : X \rightarrow V$ . Hence the divisor  $2D_4 + (D_i + E_j + E_k)$  of  $M(\mathbf{x})^2 y_i$  is anticanonical. Moreover,  $2D_4 + D_i + E_j + E_k \sim 2D_0 - D_1 - D_2 - D_3 + 2D_4$  by (7.4), thereby completing the proof.  $\square$

Now let  $C_{\text{eff}}(X)^\vee \subset \text{Hom}(\text{Pic } X \otimes \mathbb{R}, \mathbb{R})$  be the dual cone of all linear maps  $\Lambda : \text{Pic } X \otimes \mathbb{R} \rightarrow \mathbb{R}$  such that  $\Lambda([D]) \geq 0$  for every effective divisor  $D$  on  $X$ , and let  $l : \text{Hom}(\text{Pic } X \otimes \mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$  be the linear map which sends  $\Lambda$  to  $\Lambda([-K_X])$ . We then endow  $\text{Hom}(\text{Pic } X \otimes \mathbb{R}, \mathbb{R})$  with the Lebesgue measure  $ds$  normalized such that the lattice  $\text{Hom}(\text{Pic } X, \mathbb{Z})$  has covolume 1, and we endow  $H_X = l^{-1}(1)$  with the measure  $ds/dl$ . Explicitly, if  $w_0, \dots, w_4$  are coordinates for  $\text{Hom}(\text{Pic } X \otimes \mathbb{R}, \mathbb{R}) = \mathbb{R}^5$  with respect to a  $\mathbb{Z}$ -basis of  $L$  and  $l(w_0, \dots, w_4) = \alpha_0 w_0 + \dots + \alpha_4 w_4$ , then

$$\frac{ds}{dl} = dw_1 \dots \widehat{dw_i} \dots dw_5 / |\alpha_i|$$

whenever  $\alpha_i \neq 0$ .

After these preparations, we can now define  $\alpha(X)$  as

$$\alpha(X) = \int_{C_{\text{eff}}(X)^\vee \cap H_X} \frac{ds}{dl}.$$

If we let  $e_0, \dots, e_4$  be the  $\mathbb{Z}$ -basis of  $L$  with  $e_i([D_j]) = \delta_{ij}$ , then we have the following

**Lemma 7.7.**

- (a) *The hyperplane  $H_X \subset \mathbb{R}^5$  is defined by the equation  $2w_0 - w_1 - w_2 - w_3 + 2w_4 = 1$ .*
- (b) *The dual cone  $C_{\text{eff}}(X)^\vee$  is defined by the inequalities*

$$\begin{aligned} w_4 &\geq w_i \geq 0, \quad 1 \leq i \leq 3; \\ w_0 - w_i - w_j &\geq 0, \quad 1 \leq i < j \leq 3. \end{aligned}$$

*Proof.* (a) One has  $l(w_0, w_1, w_2, w_3, w_4) = \sum_{i=0}^4 e_i([-K_X])w_i$  by the definition of  $l$ . Hence  $l = 2w_0 - w_1 - w_2 - w_3 + 2w_4$  by Lemma 7.6.

(b) One has  $\sum_{i=0}^4 w_i e_i \in C_{\text{eff}}(X)^\vee$  if and only if  $\sum_{i=0}^4 w_i e_i([D]) \geq 0$  for all  $[D] \in C_{\text{eff}}(X)$ . Hence, by Proposition 7.1 we have that  $(w_0, w_1, w_2, w_3, w_4) \in C_{\text{eff}}(X)^\vee$  if and only if  $\sum_{i=0}^4 w_i e_i([D]) \geq 0$

for any  $D \in \{D_1, D_2, D_3, E_1, E_2, E_3, F_1, F_2, F_3\}$ . Now by using (7.4) and (7.5) and  $e_i([D_j]) = \delta_{ij}$ , we conclude that these nine inequalities are the same as the inequalities in the statement of the lemma.  $\square$

We are now prepared to compute the  $\alpha$ -invariant of  $X$ :

**Proposition 7.2.** *One has*

$$\alpha(X) = \frac{1}{2^6 3^2} = \frac{1}{576}.$$

*Proof.* Eliminating  $w_0 = (1 + w_1 + w_2 + w_3 - 2w_4)/2$  and then using symmetry between  $w_1, w_2, w_3$ , we obtain from Lemma 7.7 that

$$\alpha(X) = \frac{1}{2} \cdot 6 \cdot \text{vol}(Q),$$

where  $Q \subset \mathbb{R}^4$  is defined by the inequalities

$$w_4 \geq w_1 \geq w_2 \geq w_3 \geq 0 \quad \text{and} \quad w_1 + w_2 + 2w_4 \leq 1 + w_3.$$

Changing variables by the unimodular linear transformation

$$v_1 = w_1 - w_2, \quad v_2 = w_2 - w_3, \quad v_3 = w_3, \quad v_4 = w_4 - w_1,$$

we find that  $\alpha(X) = 3\text{vol}(P)$ , where  $P \subset [0, \infty)^4$  is defined by  $3v_1 + 4v_2 + 3v_3 + 2v_4 \leq 1$ . Hence,

$$\alpha(X) = \frac{1}{4!} \frac{3}{3 \cdot 4 \cdot 3 \cdot 2} = \frac{1}{2^6 3^2}.$$

$\square$

## 8. THE ADELIC VOLUME OF $X$

We keep the notation of the previous chapter. The aim of this chapter is to give an explicit description of Peyre's Tamagawa measure  $\mu_H$  on  $X(\mathbf{A}) = X(\mathbb{R}) \times \prod_p X(\mathbb{Q}_p)$ , and to compute the volume  $\mu_H(X(\mathbf{A}))$ . The interest in this comes from Peyre's prediction [19] that the constant  $c$  in the expected asymptotic formula

$$N(B) = cB(\log B)^{\text{rk Pic } X - 1}(1 + o(1))$$

should be given by  $c = \alpha(X)\mu_H(X(\mathbf{A}))$ .

**8.1. Heights and adelic metrics.** The morphism  $f : X \rightarrow V$  restricts to an isomorphism between the open subsets  $X^\circ \subset X$  and  $V^\circ \subset V$  defined by  $x_1 x_2 x_3 y_1 y_2 y_3 \neq 0$ . We conclude that

$$N(B) = |\{w \in V^\circ(\mathbb{Q}) : H(w) \leq B\}| = |\{x \in X^\circ(\mathbb{Q}) : H(f(x)) \leq B\}|$$

where the height  $H : V(\mathbb{Q}) \rightarrow \mathbb{N}$  was defined in (1.2) for a certain choice of representatives and can also be written as

$$H(\mathbf{x}, \mathbf{y}) = \prod_v \max_{1 \leq i, j \leq 3} |x_i^2 y_j|_v.$$

The aim of this section is to reinterpret this height and  $H \circ f : X(\mathbb{Q}) \rightarrow \mathbb{N}$  in terms of adelic metrics on  $\omega_V^{-1}$  and  $\omega_X^{-1}$  as in [18]. These metrics will be constructed by means of global sections on  $\omega_V^{-1}$  and  $\omega_X^{-1} = f^*(\omega_V^{-1})$ , which we obtain by glueing local sections on the open subsets  $V_{i,j} \subset V$  and  $X_{i,j} \subset X$  where  $x_i y_j \neq 0$ .

We write  $(\mathbb{P}^2 \times \mathbb{P}^2)_{i,j}$  for the open subset of  $\mathbb{P}^2 \times \mathbb{P}^2$  where  $x_i y_j \neq 0$ . On this set, we shall use affine coordinates. For  $k \neq i$  and  $l \neq j$  these are given by

$$x_k^{(i)} = \frac{x_k}{x_i} \quad \text{and} \quad y_l^{(j)} = \frac{y_l}{y_j}.$$

Then  $V_{i,j} \subset (\mathbb{P}^2 \times \mathbb{P}^2)_{i,j}$  is the affine hypersurface in  $\mathbb{A}^4$  defined by  $F_{i,j} = 0$ , where

$$F_{i,j}(x_{i+1}^{(i)}, x_{i+2}^{(i)}, y_{j+1}^{(j)}, y_{j+2}^{(j)}) = x_1^{(i)} y_2^{(j)} y_3^{(j)} + x_2^{(i)} y_1^{(j)} y_3^{(j)} + x_3^{(i)} y_1^{(j)} y_2^{(j)};$$

here and in the following we put  $x_i^{(i)} = y_j^{(j)} = 1$  and we interpret indices  $i, j, k$  in  $\mathbb{Z}/3\mathbb{Z}$ .

There is a unique global section  $s$  of  $\omega_{\mathbb{P}^2 \times \mathbb{P}^2}(D)$  which for any choice of  $i, j$  restricts to

$$s_{(\mathbb{P}^2 \times \mathbb{P}^2)_{i,j}} = \frac{dx_{i+1}^{(i)}}{x_{i+1}^{(i)}} \wedge \frac{dx_{i+2}^{(i)}}{x_{i+2}^{(i)}} \wedge \frac{dy_{j+1}^{(j)}}{y_{j+1}^{(j)}} \wedge \frac{dy_{j+2}^{(j)}}{y_{j+2}^{(j)}} \in \Gamma((\mathbb{P}^2 \times \mathbb{P}^2)_{i,j}, \omega_{\mathbb{P}^2 \times \mathbb{P}^2}(D)).$$

This can be seen directly because one has

$$\frac{dx_{i+1}^{(i)}}{x_{i+1}^{(i)}} \wedge \frac{dx_{i+2}^{(i)}}{x_{i+2}^{(i)}} = \frac{dx_{k+1}^{(k)}}{x_{k+1}^{(k)}} \wedge \frac{dx_{k+2}^{(k)}}{x_{k+2}^{(k)}}$$

on the open subset of  $\mathbb{P}^2$  where  $x_i x_k \neq 0$ . Alternatively, this claim is a special case of a general result for toric varieties (see [4, Lemma 12]). The latter result also shows that  $s$  is a global generator of the  $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}$ -module  $\omega_{\mathbb{P}^2 \times \mathbb{P}^2}(D)$ .

Now put  $F = x_1 y_2 y_3 + x_2 y_1 y_3 + x_3 y_1 y_2$ , and then define

$$(8.1) \quad \omega_{i,j} = \frac{x_1 x_2 x_3 y_1 y_2 y_3}{x_i^2 y_j^2 F} s \in \Gamma((\mathbb{P}^2 \times \mathbb{P}^2), \omega_{\mathbb{P}^2 \times \mathbb{P}^2}(V + 2H_{x_i} + H_{y_i})),$$

where  $H_{x_i}$  (resp.  $H_{y_i}$ ) are the prime divisors on  $\mathbb{P}^2 \times \mathbb{P}^2$  defined by the vanishing of  $x_i$  (resp.  $y_j$ ). Then,  $\omega_{i,j}$  is a global generator of  $\omega_{\mathbb{P}^2 \times \mathbb{P}^2}(V + 2H_{x_i} + H_{y_i})$  with

$$\omega_{i,j} = \frac{1}{F_{i,j}} dx_{i+1}^{(i)} \wedge dx_{i+2}^{(i)} \wedge dy_{j+1}^{(j)} \wedge dy_{j+2}^{(j)}$$

on  $(\mathbb{P}^2 \times \mathbb{P}^2)_{i,j}$ .

We now consider the Poincaré residue map  $\text{Res}: \omega_{\mathbb{P}^2 \times \mathbb{P}^2}(V) \rightarrow \iota_* \omega_V$  for the inclusion map  $\iota: V \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$ . The Poincaré residue map is usually given as a homomorphism  $\Omega_W^n(V) \rightarrow \iota_* \Omega_V^{n-1}$  for the inclusion map  $\iota: V \rightarrow W$  of a non-singular hypersurface  $V \subset W$  in an  $n$ -dimensional non-singular variety (cf. [Re3, p. 89], for example). More generally, one can also use Poincaré residues to define local sections on the canonical sheaf  $\omega_V$  of an arbitrary normal hypersurface  $V \subset W$  (cf. [We]) as one still gets regular  $(n-1)$ -forms on the non-singular locus  $V_{ns}$  of  $V$  and since  $\omega_V = j_* \Omega_{V_{ns}}^{n-1}$  for the inclusion map  $j: V_{ns} \rightarrow V$ . After these general remarks we return to our specific situation.

By regarding  $\omega_{i,j}$  as a local section of  $\omega_{\mathbb{P}^2 \times \mathbb{P}^2}(V)$  on  $(\mathbb{P}^2 \times \mathbb{P}^2)_{i,j}$  we obtain a local section  $\text{Res}(\omega_{i,j}) \in \Gamma(V_{i,j}, \omega_V)$ , where

$$\text{Res}(\omega_{i,j}) = (-1)^{\text{pos}(z)+1} \frac{1}{\partial F_{i,j} / \partial z} dx_{i+1}^{(i)} \wedge \dots \widehat{dz} \dots \wedge dy_{j+2}^{(j)},$$

on the open subset of  $V_{i,j}$  where  $\partial F_{i,j} / \partial z \neq 0$  and  $\text{pos}(z) \in \{1, 2, 3, 4\}$  is the position of  $z \in \{x_{i+1}^{(i)}, x_{i+2}^{(i)}, y_{j+1}^{(j)}, y_{j+2}^{(j)}\}$ . This defines  $\text{Res}(\omega_{i,j})$  on the non-singular locus  $U_{i,j}$  of  $V_{i,j}$  with  $\text{Res}(\omega_{i,j}) \neq 0$  everywhere on  $U_{i,j}$ . As  $V_{i,j}$  is normal, we may then extend  $\text{Res}(\omega_{i,j})$  to a volume form on  $V_{i,j}$  by a standard argument (see [13, p. 181]).

Hence there is an inverse nowhere vanishing local section  $\tau_{i,j} = \text{Res}(\omega_{i,j})^{-1} \in \Gamma(V_{i,j}, \omega_V^{-1})$  with

$$\tau_{i,j} = (-1)^{\text{pos}(z)+1} \partial F_{i,j} / \partial z \frac{\partial}{\partial x_{i+1}^{(i)}} \wedge \dots \widehat{\frac{\partial}{\partial z}} \dots \wedge \frac{\partial}{\partial y_{j+2}^{(j)}}$$

on the non-singular locus of  $V_{i,j}$ .

We shall also write  $\sigma_{i,j} \in \Gamma(X_{i,j}, \omega_X^{-1})$  for the local section corresponding to

$$f^* \tau_{i,j} := \tau_{i,j} \otimes 1 \in \Gamma(f^{-1}(V_{i,j}), f^{-1} \omega_V^{-1} \otimes_{f^{-1} \mathcal{O}_V} \mathcal{O}_X) = \Gamma(X_{i,j}, f^* \omega_V^{-1}).$$

As  $\tau_{i,j} \in \Gamma(V_{i,j}, \omega_V^{-1})$  is inverse to the volume form  $\text{Res}(\omega_{i,j})$  on  $V_{i,j}$ , we conclude that  $\sigma_{i,j}$  is inverse to the volume form  $\sigma_{i,j}^{-1}$  on  $X_{i,j}$  corresponding to

$$f^*(\text{Res}(\omega_{i,j})) = \text{Res}(\omega_{i,j}) \otimes 1 \in \Gamma(f^{-1}(V_{i,j}), f^{-1} \omega_V \otimes_{f^{-1} \mathcal{O}_V} \mathcal{O}_X) = \Gamma(X_{i,j}, f^* \omega_V).$$

**Lemma 8.1.** *Let  $i, j, k, l \in \{1, 2, 3\}$ .*

- (a) *We have  $\tau_{i,j} = (x_i^{(k)})^2 y_j^{(l)} \tau_{k,l}$  on  $V_{i,j} \cap V_{k,l}$ .*
- (b) *We have  $\sigma_{i,j} = (x_i^{(k)})^2 y_j^{(l)} \sigma_{k,l}$  on  $X_{i,j} \cap X_{k,l}$ .*

*Proof.* (a) By (8.1) we have  $\omega_{i,j} = (x_k^{(i)})^2 y_l^{(j)} \omega_{k,l}$  on  $V_{i,j} \cap V_{k,l}$ . Hence  $\text{Res}(\omega_{i,j}) = (x_k^{(i)})^2 y_l^{(j)} \text{Res}(\omega_{k,l})$  and  $\tau_{i,j} = (x_i^{(k)})^2 y_j^{(l)} \tau_{k,l}$  on  $V_{i,j} \cap V_{k,l}$ .

(b) Let  $\bar{a} \in \Gamma(X_{i,j} \cap X_{k,l}, O_X)$  be the image of  $a \in \Gamma(X_{i,j} \cap X_{k,l}, O_X) = \Gamma(X_{i,j} \cap X_{k,l}, f^{-1}O_V)$  under the natural map from  $f^{-1}O_V$  to  $O_X$ . Then,

$$f^*(a\tau_{k,l}) = a\tau_{k,l} \otimes 1 = \bar{a}(\tau_{k,l} \otimes 1) = \bar{a}f^*(\tau_{k,l})$$

on  $X_{i,j} \cap X_{k,l}$ . Hence  $f^*(\tau_{i,j}) = (x_i^{(k)})^2 y_j^{(l)} f^*(\tau_{k,l})$  on  $X_{i,j} \cap X_{k,l}$  by (a), thereby proving the assertion in (b).  $\square$

The lemma implies that  $\tau_{i,j} \in \Gamma(V_{i,j}, \omega_V^{-1})$  extends to a global anticanonical section that we still denote by  $\tau_{i,j} \in \Gamma(V, \omega_V^{-1})$ . Similarly, we let  $\sigma_{i,j}$  be the global anticanonical section on  $X$  defined by  $\sigma_{i,j} = f^*\tau_{i,j}$ .

For each place  $v$  of  $\mathbb{Q}$ , the global sections  $\tau_{i,j}$ ,  $1 \leq i, j \leq 3$ , define a  $v$ -adic norm on  $\omega_V^{-1}$  with

$$(8.2) \quad \|\tau(w_v)\|_v = \min_{i,j} \left| \frac{\tau}{\tau_{i,j}}(w_v) \right|_v = \min_{i,j} |\tau \text{Res}(\omega_{i,j})|_v$$

for a local section  $\tau$  of  $\omega_V^{-1}$  defined at  $w_v \in V(\mathbb{Q}_v)$ . Here the minimum is taken over all  $i, j \in \{1, 2, 3\}$  such that  $\tau_{i,j}(w_v) \neq 0$ . This definition is similar to the definition in [18, pp. 107-108], although it is called a  $v$ -adic metric there.

In the same way we may define a  $v$ -adic norm on  $\omega_X^{-1}$  by letting

$$(8.3) \quad \|\sigma(x_v)\|_v = \min_{i,j} \left| \frac{\sigma}{\sigma_{i,j}}(x_v) \right|_v.$$

for a local section  $\sigma$  of  $\omega_X^{-1}$  defined at  $x_v \in X(\mathbb{Q}_v)$ . Here now the minimum is taken over all  $i, j \in \{1, 2, 3\}$  such that  $\sigma_{i,j}(x_v) \neq 0$ . We then have, just as in [4, Lemma 15], the following result.

**Lemma 8.2.**

- (a) *Let  $w \in V(\mathbb{Q})$  and  $\tau$  be a local section of  $\omega_V^{-1}$  with  $\tau(w) \neq 0$ . Then  $H(w) = \prod_v \|\tau(w)\|_v^{-1}$ .*
- (b) *Let  $x \in X(\mathbb{Q})$  and  $\sigma$  be a local section of  $\omega_X^{-1}$  with  $\sigma(x) \neq 0$ . Then  $H(f(x)) = \prod_v \|\sigma(x)\|_v^{-1}$ .*

*Proof.* On applying the product formula  $\prod_v |\alpha|_v = 1$  for  $\alpha \in \mathbb{Q}^*$ , it suffices in both cases to prove the formula for one local section. To prove (a), suppose that  $w \in V_{k,l}$  and let  $\tau = \tau_{k,l}$ . Then  $\tau(w) \neq 0$ , and by (8.2) and Lemma 8.1(a) we see that

$$\|\tau(w)\|_v^{-1} = \max_{i,j} \left| \frac{\tau_{i,j}}{\tau_{k,l}}(w_v) \right|_v = \frac{1}{|x_k^2 y_l|_v} \max_{i,j} |x_i^2 y_j|_v$$

holds for each place  $v$ . Hence the desired identity  $\prod_v \|\tau(w)\|_v^{-1} = H(w)$  follows from the product formula.

To prove (b), we may assume that  $x \in X_{k,l}$  and choose  $\sigma = \sigma_{k,l}$ . The proof is then the same as for (a), but based on using (8.3) and Lemma 8.1(b).  $\square$

**8.2. The volume of the adelic space  $X(\mathbf{A})$ .** We now describe Peyre's Tamagawa measure  $\mu_H$  on  $X(\mathbf{A}) = X(\mathbb{R}) \times \prod_p X(\mathbb{Q}_p)$  defined by the adelic metric on  $\omega_X^{-1}$  of all  $v$ -adic norms in (8.3), and compute the volume of the adelic space  $X(\mathbf{A})$  with respect to this measure.

To obtain this measure, we recall the definition in [18, (2.2.1)] of a measure  $\mu_v$  on  $X(\mathbb{Q}_v)$  associated to a  $v$ -adic norm on  $\omega_X^{-1}$ . Let  $|\sigma_{i,j}^{-1}|_v$  be the  $v$ -adic density on  $X_{i,j}(\mathbb{Q}_v)$  of the volume form  $\sigma_{i,j}^{-1}$  on

$X_{i,j}$ . Then, for a Borel subset  $N_v$  of  $X_{i,j}(\mathbb{Q}_v)$ , and with the  $v$ -adic norm on  $\omega_X^{-1}$  defined in (8.3), we put

$$\mu_v(N_v) = \int_{N_v} \frac{|\sigma_{i,j}^{-1}|_v}{\max_{k,l} |\sigma_{k,l} \sigma_{i,j}^{-1}|_v}.$$

This defines a measure  $\mu_v$  on  $X(\mathbb{Q}_v)$ . On applying Lemma 8.1(b), we may rewrite this as

$$\mu_v(N_v) = \int_{N_v} \frac{|\sigma_{i,j}^{-1}|_v}{\max_{k,l} |(x_k^{(i)})^2 y_l^{(j)}|_v}.$$

As usual, we write  $\mu_\infty = \mu_v$  when  $\mathbb{Q}_v = \mathbb{R}$  and  $\mu_p = \mu_v$  when  $\mathbb{Q}_v = \mathbb{Q}_p$ .

**Lemma 8.3.** *Let*

$$D = \left\{ w \in V^\circ(\mathbb{R}) : |x_1| \leq |x_3|, |x_2| \leq |x_3|, |y_1| \geq |y_2|, x_1^{(3)} > 0, y_1^{(3)} > 0 \right\}.$$

*Then*

$$\mu_\infty(X(\mathbb{R})) = 24 \int_D \frac{|\text{Res}(dx_1^{(3)} \wedge dx_2^{(3)} \wedge dy_1^{(3)} \wedge dy_2^{(3)})|}{\max(y_1^{(3)}, 1)}.$$

*Proof.* As the inverse image  $f^*(\tau_{i,j}^{-1})$  of the volume form  $\tau_{i,j}^{-1} = \text{Res}(\omega_{i,j})$  on  $V_{i,j}$  is sent to the volume form  $\sigma_{i,j}^{-1}$  on  $X_{i,j}$  under the canonical map from  $f^*\omega_V \rightarrow \omega_X$ , we have thus for Borel subsets  $N \subset X^\circ(\mathbb{R})$  that

$$(8.4) \quad \mu_\infty(N) = \int_{f(N)} \frac{|\text{Res}(\omega_{i,j})|}{\max_{k,l} |\sigma_{k,l} \text{Res}(\omega_{i,j})|} = \int_{f(N)} \frac{|\text{Res}(\omega_{i,j})|}{\max_{k,l} |(x_k^{(i)})^2 y_l^{(j)}|}.$$

for any fixed  $i, j \in \{1, 2, 3\}$ .

The hyperoctahedral group  $\mathbb{Z}_2 \wr S_3$  of order  $2^3 \times 3!$  acts on the affine hypersurface in  $\mathbb{A}^6$  defined by  $F = 0$ . This group consists of signed symmetries over  $\varrho \in S_3$  sending  $(x_i, y_i)$  to one of  $(x_{\varrho(i)}, y_{\varrho(i)})$  or  $-(x_{\varrho(i)}, y_{\varrho(i)})$  for  $i \in \{1, 2, 3\}$  and we obtain in this way an action of  $\mathbb{Z}_2 \wr S_3$  on  $V$  as well. As the symmetry sending all  $(\mathbf{x}, \mathbf{y})$  to  $-(\mathbf{x}, \mathbf{y})$  is trivial on  $V$ , we get in fact a faithful action of the octahedral group  $O$  of order 24 on  $V$ , which preserves  $V^\circ$ . The set  $D$  is a fundamental domain for the (measure-preserving) action of this group, hence  $\mu_\infty(V^\circ(\mathbb{R})) = 24\mu_\infty(D)$ .

We now apply (8.4) with  $i = j = 3$ . Then  $\omega_{3,3} = dx_1^{(3)} \wedge dx_2^{(3)} \wedge dy_1^{(3)} \wedge dy_2^{(3)}$  and

$$\max_{1 \leq k \leq 3} |(x_k^{(3)})^2| \max_{1 \leq l \leq 3} |y_l^{(3)}| = \max(|y_1^{(3)}|, 1)$$

on  $D$ . Hence

$$\mu_\infty(D) = \int_D \frac{|\text{Res}(dx_1^{(3)} \wedge dx_2^{(3)} \wedge dy_1^{(3)} \wedge dy_2^{(3)})|}{\max(|y_1^{(3)}|, 1)}$$

and we are done.  $\square$

We are now prepared to compute  $\mu_\infty(X(\mathbb{R}))$  explicitly. This is the counterpart to Lemma 2.10.

**Lemma 8.4.** *We have  $\mu_\infty(X(\mathbb{R})) = 96 \log 2 - 12 + 4\pi^2$ .*

*Proof.* Set  $t_1 = x_1^{(3)}$ ,  $t_2 = x_2^{(3)}$ ,  $u_1 = y_1^{(3)}$  and  $u_2 = y_2^{(3)}$ . Then  $F_{3,3} = t_1 u_2 + t_2 u_1 + u_1 u_2$  and

$$|\text{Res}(\omega_{3,3})| = \frac{dt_1 du_1 dt_2}{|\partial F_{3,3} / \partial u_2|} = \frac{dt_1 du_1 dt_2}{|t_1 + u_1|}.$$

Moreover, we have the equivalences

$$|y_1| \geq |y_2| \iff |u_1| \geq |u_2| \iff |t_2| \leq |t_1 + u_1|$$

as  $-u_1 t_2 = (t_1 + u_1) u_2$  on  $V$ . By the previous lemma we conclude that

$$\begin{aligned} \frac{\mu_\infty(X(\mathbf{R}))}{24} &= \int_0^\infty \int_{-1}^1 \int_{t_2=0}^{\min(1, |t_1+u_1|)} \frac{dt_2}{|t_1+u_1|} \frac{dt_1 du_1}{\max(u_1, 1)} \\ &= \int_0^\infty \int_{-1}^1 \min\left(\frac{1}{|t+u|}, 1\right) \frac{dt du}{\max(u, 1)} \\ &= \int_0^2 \frac{2-u+\log(u+1)}{\max(u, 1)} du + \int_2^\infty \log\left(\frac{u+1}{u-1}\right) \frac{du}{u}, \end{aligned}$$

and a straightforward computation now shows that this quantity equals  $4 \log 2 - \frac{1}{2} + \frac{\pi^2}{6}$ , as desired.  $\square$

To compute the  $p$ -adic volumes  $\mu_p(X(\mathbb{Q}_p))$ , we shall make use of the scheme  $\underline{X} \subset \mathbb{P}_{\mathbb{Z}}^2 \times \mathbb{P}_{\mathbb{Z}}^2 \times \mathbb{P}_{\mathbb{Z}}^2$  with coordinates  $(\mathbf{x}, \mathbf{y}, \mathbf{z})$  defined by the equations  $x_1 z_1 + x_2 z_2 + x_3 z_3 = 0$  and  $y_1 z_1 = y_2 z_2 = y_3 z_3$ . It is smooth over  $\mathbb{Z}$ , and there is an extension of  $f : X \rightarrow V$  to a morphism  $\underline{f} : \underline{X} \rightarrow \underline{V}$  with  $\underline{f}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{x}, \mathbf{y})$  onto the subscheme  $\underline{V} \subset \mathbb{P}_{\mathbb{Z}}^2 \times \mathbb{P}_{\mathbb{Z}}^2$  defined by  $x_1 y_2 y_3 + x_2 y_1 y_3 + x_3 y_1 y_2 = 0$ . There is also a functorial homomorphism  $\underline{f}^* \omega_{\underline{V}/\mathbb{Z}} \rightarrow \omega_{\underline{X}/\mathbb{Z}}$  of invertible  $\mathcal{O}_X$ -modules for the relative canonical (or dualising) invertible sheaves  $\omega_{\underline{V}/\mathbb{Z}}$  and  $\omega_{\underline{X}/\mathbb{Z}}$ , which must be an isomorphism as  $f$  and the base extensions  $\underline{f}_{\mathbb{F}_p} : \underline{X}_{\mathbb{F}_p} \rightarrow \underline{V}_{\mathbb{F}_p}$  are crepant (see [4, Theorem 4]).

Now consider the dual isomorphism from  $\omega_{\underline{X}/\mathbb{Z}}^{-1}$  to  $\underline{f}^* \omega_{\underline{V}/\mathbb{Z}}^{-1}$ . It extends the isomorphism from  $\omega_X^{-1}$  to  $f^* \omega_V^{-1}$  that was used to define  $\sigma_{i,j} = f^* \tau_{i,j}$ . We may thus extend  $\sigma_{i,j}$  to global sections  $\underline{\sigma}_{i,j} = \underline{f}^* \underline{\tau}_{i,j}$  of  $\omega_{\underline{X}/\mathbb{Z}}^{-1} = \underline{f}^* \omega_{\underline{V}/\mathbb{Z}}^{-1}$  as follows. We first define  $\underline{\tau}_{i,j}$  and  $\underline{\sigma}_{i,j}$  on the principal open subsets where  $x_i y_j \neq 0$  in the same way as we defined  $\sigma_{i,j}$  and  $\tau_{i,j}$ , and we then use an analogue of Lemma 8.1 for  $\underline{V}$  and  $\underline{X}$  to extend these sections to global sections.

**Lemma 8.5.** *For all primes  $p$  one has*

$$\mu_p(X(\mathbb{Q}_p)) = 1 + \frac{5}{p} + \frac{5}{p^2} + \frac{1}{p^3}.$$

**Proof.** In [21, Def. 2.9] there is defined a measure  $m_p$  on  $\underline{X}(\mathbb{Z}_p)$  called the model measure for which  $m_p(\underline{X}(\mathbb{Z}_p)) = |\underline{X}(\mathbb{F}_p)|/p^{\dim X}$  (see [21, Cor. 2.15]). As  $\underline{X} \times_{\mathbb{Z}} \mathbb{F}_p$  is a  $\mathbb{P}^1$ -bundle over a del Pezzo surface  $B$  of degree 6 over  $\mathbb{F}_p$ , we conclude that  $|\underline{X}(\mathbb{F}_p)| = (p^2 + 4p + 1)(p + 1)$  and

$$m_p(\underline{X}(\mathbb{Z}_p)) = 1 + \frac{5}{p} + \frac{5}{p^2} + \frac{1}{p^3}.$$

As  $\underline{X}$  is proper over  $\mathbb{Z}$ , there is a natural bijection  $\underline{X}(\mathbb{Z}_p) = X(\mathbb{Q}_p)$ . To complete the proof of the lemma, it is thus enough to show that  $m_p = \mu_p$ . The definitions of  $m_p$  and  $\mu_p$  are both based on Peyre's construction [18, (2.2.1)] of a measure on  $X(\mathbb{Q}_v)$  associated to a  $v$ -adic norm on  $\omega_X^{-1}$ . For  $m_p$  one uses a  $p$ -adic norm  $\|\cdot\|_p^*$  called the model norm, as described in [21, (2.9)]. Thus it only remains to prove that this norm coincides with the  $p$ -adic norm  $\|\cdot\|_p$  in (8.3) used to define  $\mu_p$ .

Therefore, let  $\sigma$  be a local section of  $\omega_X^{-1}$  defined at  $x_p \in X(\mathbb{Q}_p)$ . To show that  $\|\sigma\|_p^* = \|\sigma\|_p$  in a neighbourhood of  $x_p$ , we choose  $i, j$  such that  $x_p \in X_{i,j}(\mathbb{Q}_p)$ . The restriction of  $\omega_X^{-1}$  to  $U = X_{i,j}$  is a free  $\mathcal{O}_U$ -module generated by  $\sigma_{i,j}$  as  $\sigma_{i,j}$  is the inverse to the volume form  $f^*(\text{Res}(\omega_{i,j}))$  on  $X_{i,j}$ . By the same argument one obtains that the restriction of  $\omega_{\underline{X}/\mathbb{Z}}^{-1}$  to  $\underline{U} = \underline{X}_{i,j}$  is a free  $\mathcal{O}_{\underline{U}}$ -module generated by  $\underline{\sigma}_{i,j}$ . As  $\underline{\sigma}_{i,j}$  restricts to  $\sigma_{i,j}$  on  $X_{i,j}$ , we conclude from the definition of the model norm (see [21, 1.9 and 2.9]) that  $\|\sigma_{i,j}\|_p^* = 1$  on  $X_{i,j}$ , and by (8.3) that  $|\sigma_{i,j}|_p = 1$  on  $X_{i,j}$ . Hence

$$\|\sigma\|_p^* = |\sigma/\sigma_{i,j}|_p \|\sigma_{i,j}\|_p^* = |\sigma/\sigma_{i,j}|_p \|\sigma_{i,j}\|_p = \|\sigma\|_p$$

in a neighbourhood of  $x_p$ , as was to be shown.

Now let

$$L_p(s, \text{Pic } \overline{X}) = \det(1 - p^{-s} \text{Fr}_p \mid \text{Pic } (X_{\overline{\mathbb{F}_p}}) \otimes \mathbb{Q})^{-1}.$$



Then, as  $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$  acts trivially on  $\text{Pic}(X_{\overline{\mathbb{F}}_p}) \cong \mathbb{Z}^5$ , we conclude that

$$L_p(s, \text{Pic } \overline{X}) = (1 - p^{-s})^{-5},$$

so that

$$L(s, \text{Pic } \overline{X}) = \prod_{\text{all } p} L_p(s, \text{Pic } \overline{X}) = \zeta(s)^5$$

In particular,

$$\lim_{s \rightarrow 1} (s-1)^5 L(s, \text{Pic } \overline{X}) = 1 \quad \text{and} \quad L_p(1, \text{Pic } \overline{X})^{-1} = \left(\frac{p-1}{p}\right)^5.$$

Peyre's measure  $\mu_H$  on  $X(\mathbf{A}) = X(\mathbb{R}) \times \prod_p X(\mathbb{Q}_p)$  is therefore given by

$$\mu_H = \mu_\infty \times \prod_p \left(\frac{p-1}{p}\right)^5 \mu_p,$$

and it is shown in [19, Def. 4.6] that this gives a well defined measure on  $X(\mathbf{A})$ . As  $X(\mathbb{Q})$  is dense in  $X(\mathbf{A})$ , we conclude (see [19, Def. 4.8]) that

$$(8.5) \quad \mu_H(X(\mathbf{A})) = \mu_\infty(X(\mathbb{R})) \prod_p \left(\frac{p-1}{p}\right)^5 \mu_p(X(\mathbb{Q}_p)).$$

We may now combine the conclusions of Lemma 8.4, Lemma 8.5 and (8.5) to infer the following result.

**Proposition 8.1.** *Peyre's Tamagawa constant  $\tau_H(X) = \mu_H(X(\mathbf{A}))$  associated to the adelic metric of all  $v$ -adic norms in (8.3) is given by*

$$\tau_H(X) = (96 \log 2 - 12 + 4\pi^2) \prod_p \left(1 + \frac{5}{p} + \frac{5}{p^2} + \frac{1}{p^3}\right).$$

**8.3. The leading term of the asymptotic formula.** We finally show that the asymptotic formula for  $N(B)$  in Theorem 1.1 is in accordance with conjectures made in [19]. As the biprojective threefold  $V$  defined by (1.1) is singular, we cannot refer to the original conjectures of Manin [10] and Peyre [18]. To overcome this, we make use of the observation in Section 8.1 that

$$N(B) = \{x \in X^\circ(\mathbb{Q}) : (H \circ f)(x) \leq B\}.$$

We have also seen in (8.3) that

$$H(f(x)) = \prod_v \|\sigma(x)\|_v^{-1}$$

for a local anticanonical section  $\sigma$  with  $\sigma(x) \neq 0$ . We may therefore refer to the conjectures of Peyre [19] for “almost” Fano varieties instead. The following result shows that  $X$  satisfies the three conditions for being such a variety.

**Lemma 8.6.** *Let  $X \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$  be as before. Then*

- (a)  $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ ;
- (b) *The geometric Picard group  $\text{Pic}(X_{\overline{\mathbb{Q}}})$  is torsion-free;*
- (c) *The anticanonical class is in the interior of  $C_{\text{eff}}(X)$ .*

*Proof.* To prove (a), we apply the Leray spectral sequence  $H^i(Y, R^j p_* \mathcal{O}_X) \implies H^{i+j}(X, \mathcal{O}_X)$  to the  $\mathbb{P}^1$ -bundle  $p : X \rightarrow Y$ . Then, we obtain isomorphisms  $H^i(Y, \mathcal{O}_Y) = H^i(X, \mathcal{O}_X)$  for all  $i$ , with  $H^1(Y, \mathcal{O}_Y) = H^2(Y, \mathcal{O}_Y) = 0$  for a del Pezzo surface.

For (b), we use that  $X$  is a  $\mathbb{P}^1$ -bundle over a del Pezzo surface  $Y$  of degree 6. This gives  $\text{Pic}(X_{\overline{\mathbb{Q}}}) \cong \text{Pic}(Y_{\overline{\mathbb{Q}}}) \oplus \mathbb{Z} \cong \mathbb{Z}^5$ .

Finally (c) follows from Lemma 7.7 and the fact that  $3D_i + E_j + E_k + F_i$  is an anticanonical divisor if  $\{i, j, k\} = \{1, 2, 3\}$  (see Lemma 7.6).  $\square$



If we implicitly assume that there are no accumulating subvarieties on  $X$  outside  $X \setminus X^\circ$ , then Peyre’s “empiric formula” in [19, (5.1)] for an almost Fano variety  $X$  suggests that

$$\{x \in X^\circ(\mathbf{Q}) : H(f(x)) \leq B\} \sim \Theta_H(X) B(\log B)^{r-1},$$

where  $r = \text{rk Pic } X$  and  $\Theta_H(X) = \alpha(X)\tau_H(X)$ . As  $r = 1 + \text{rk Pic } Y = 5$  and

$$\Theta_H(X) = \frac{\pi^2 - 3 + 24 \log 2}{144} \prod_p \left(1 + \frac{5}{p} + \frac{5}{p^2} + \frac{1}{p^3}\right)$$

by Propositions 7.2 and 8.1, the asymptotic formula in Theorem 1.1 is therefore of the form predicted by Peyre [19].

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